Fachbereich Mathematik der Universität Hamburg **SoSe 2022**
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Differential Equations II for Engineering Students

Work sheet 5

Exercise 1:

Using a suitable product ansatz, solve the following Dirichlet boundary value problem for the Laplace equation on the circle $r^2 = x^2 + y^2 \leq 9$.

$$\begin{aligned} &r^2 u_{rr} + r u_r + u_{\varphi\varphi} = 0 && 0 \leq r < 3 \\ &u(3,\varphi) = \cos^2(\varphi) && \varphi \in \mathbb{R}. \end{aligned}$$

Hints:

To solve Euler's equation
$r^2 \cdot g''(r) + ar \cdot g'(r) + b \cdot g(r) = 0$ use the ansatz $g(r) = r^k$.
It holds: $\cos^2(\varphi) = \dfrac{1}{2}\left(1 + \cos(2\varphi)\right)$.

Exercise 2:

Determine the solution to the initial boundary value problem (IBVP)

$$\begin{aligned} u_t - u_{xx} &= \sin(x)\, t && 0 < x < \pi\,,\ 0 < t\,, \\ u(x,0) &= 4\sin(3x) + \tfrac{x}{\pi} && 0 \leq x \leq \pi, \\ u(0,t) &= \phi_1(t) = 0 && 0 \leq t, \\ u(\pi,t) &= \phi_2(t) = 1 && 0 \leq t. \end{aligned}$$

Note: First homogenize the boundary conditions by using the function

$$v(x,t) = u(x,t) - \phi_1(t) - \frac{x-a}{b-a}\left(\phi_2(t) - \phi_1(t)\right)$$

with $a = 0$ and $b = \pi$ and then replace the $u-$expressions with corresponding $v-$expressions. You get e.g.

$$u_t = v_t + \dot{\phi}_1 + \frac{x-a}{b-a}\left(\dot{\phi}_2 - \dot{\phi}_1\right).$$