Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
Prof. Dr. J. Behrens
Dr. H. P. Kiani
S. Onyshkevych

Differential Equations II for Engineering Students

Homework sheet 6

Exercise 1:

a) Solve the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx}, && \text{on } \mathbb{R}^2\,, \\ u(x,0) &= 2\sin(4\pi x) && x \in \mathbb{R}\,, \\ u_t(x,0) &= \cos(\pi x) && x \in \mathbb{R}\,. \end{aligned}$$

b) Given the problem

$$u_{tt} = 9u_{xx}, \quad \text{for } x \in \mathbb{R},\ t > 0\,,$$

$$u(x,0) = f(x) = \begin{cases} 2 & -1 \le x \le 1\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_t(x,0) = 0.$$

Sketch the obtained solution using d'Alembert's formula for

$t = 0,\ \frac{1}{6},\ \frac{1}{3},\ \frac{2}{3},\ 1\,.$

Exercise 2:

The following problem is given for $u(x,y,t)$.

$$\begin{aligned} u_t &= u_{xx} + u_{yy}, && x, y \in (0,\pi),\ t > 0, \\ u(0,y,t)) &= u(\pi,y,t) = 0, && \text{for } y \in (0,\pi),\ t > 0, \\ u(x,0,t)) &= u(x,\pi,t) = 0, && \text{for } x \in (0,\pi),\ t > 0, \\ u(x,y,0) &= \frac{1}{2}\left(\sin(2x) + \sin(x)\right)\sin(y) && \text{for } x, y \in (0,\pi). \end{aligned}$$

a) Using the ansatz $u(x,y,t) = T(t) \cdot X(x) \cdot Y(y)$ for the solution of the differential equation, derive three decoupled ordinary differential equations for X, Y and T.

b) Derive first from the boundary values

$$\begin{aligned} u(0,y,t) = u(\pi,y,t) &= 0, && \text{for } y \in [0,\pi],\ t > 0, \\ u(x,0,t) = u(x,\pi,t) &= 0, && \text{for } x \in [0,\pi],\ t > 0, \end{aligned}$$

the boundary conditions for the solutions of the differential equations for X and Y, and solve the obtained ordinary boundary value problems for X and Y.

Then determine the appropriate functions $T(t)$.

c) Determine a series representation of the solution u to the original problem and fit it to the initial values

$$u(x, y, 0) = \frac{1}{2}\left(\sin(2x) + \sin(x)\right)\sin(y) \quad \text{for } x, y \in [0, \pi].$$

How does the solution behave for $t \to \infty$?