Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Work sheet 6

Exercise 1:

From Lecture 9 we know d'Alembert's formula

$$\hat{u}(x,t) = \frac{1}{2}\,(f(x+ct)+f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\,d\alpha$$

for solving the initial value problem for the (homogeneous) wave equation

$$\hat{u}_{tt} - c^2\hat{u}_{xx} = 0,\ \hat{u}(x,0) = f(x),\ \hat{u}_t(x,0) = g(x),\ x \in \mathbb{R},\, c > 0.$$

The function

$$\tilde{u}(x,t) \;=\; \frac{1}{2c}\int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\,\mathrm{d}\omega\mathrm{d}\tau \tag{1}$$

solves the following initial value problem

$$\tilde{u}_{tt} \;-\; c^2\tilde{u}_{xx} \;=\; h(x,t) \qquad \tilde{u}(x,0) \;=\; \tilde{u}_t(x,0) \;=\; 0\,. \tag{2}$$

(Proof: Leibniz formula for the derivation of parameter-dependent integrals)

The initial value problem is to be solved

$$u_{tt} \;-\; 4u_{xx} \;=\; 6x\sin t, \qquad x \in \mathbb{R},\, t > 0$$
$$u(x,0) \;=\; x,\ x \in \mathbb{R}, \qquad u_t(x,0) \;=\; \sin(x),\ x \in \mathbb{R}$$

a) Compute the solution $\hat{u}$ to the IVP

$$\hat{u}_{tt} \;-\; 4\hat{u}_{xx} \;=\; 0, \qquad x \in \mathbb{R},\, t > 0$$
$$\hat{u}(x,0) \;=\; x,\ x \in \mathbb{R}, \qquad \hat{u}_t(x,0) \;=\; \sin(x),\ x \in \mathbb{R}.$$

b) Compute the solution $\tilde{u}$ to the IVP

$$\tilde{u}_{tt} \;-\; 4\tilde{u}_{xx} \;=\; 6x\sin t, \qquad x \in \mathbb{R},\, t > 0$$
$$\tilde{u}(x,0) \;=\; 0,\ x \in \mathbb{R}, \qquad \tilde{u}_t(x,0) \;=\; 0,\ x \in \mathbb{R}$$

c) By inserting u into the differential equation and checking the initial values, show that $u = \tilde{u} + \hat{u}$ solves the initial value problem (2).

Exercise 2:

a) Using a product ansatz, derive the series representation given in lecture 10 (page 18) for the solution of the following Neumann problem.

$$\begin{aligned} u_t &= u_{xx}, & 0 < x < 1,\, t > 0, \\ u(x,0) &= g(x), & 0 < x < 1, \\ u_x(0,t) &= u_x(1,t) = 0 & t > 0. \end{aligned}$$

b) Solve the initial boundary value problem a) with $g(x) = 2\pi x - \sin(2\pi x)$.

Hint: $2\sin(\alpha)\cdot\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$.