

# Differential Equations II



Introduction

# Your Professor

## Coordinates:



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Office hours after appointment by email

## Background

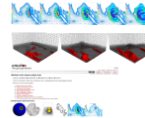


## Short CV

Since 2009 Prof. @ Uni Hamburg, Dept. Mathematics  
2006-2009 Lead Tsunami Group @ AWI, Adjunct @ Uni Bremen  
2005 Habilitation (Mathematics) @ TUM  
2003-2004 Visiting Scientist @ NCAR, Boulder, CO, USA  
1998-2006 Assistant Prof (Wiss. Rat) @ TUM, Scient. Computing  
1996-1998 Post-Doc @ AWI  
1991-1996 Dr. rer. nat. (Mathematics) @ AWI/Uni Bremen  
1991 Diploma Mathematics @ Uni Bonn

## Research Interests

Adaptive Tsunami Modeling  
Adaptive Atmospheric Modeling  
Mesh Generation  
Multi-Scale Simulation



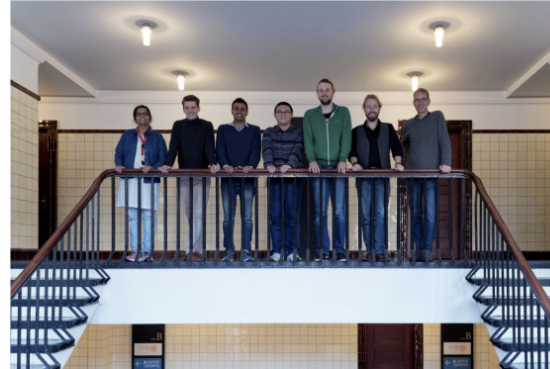
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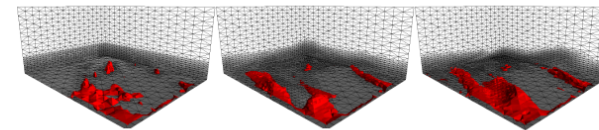
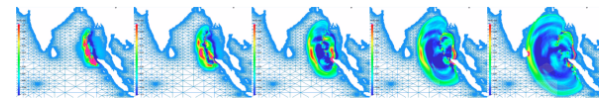
# Research Interests

Adaptive Tsunami Modeling

Adaptive Atmospheric Modeling

Mesh Generation

Multi-Scale Simulation



**amafos**  
the grid generator

Welcome to the amafos project home

amafos is an Adaptive Mesh generator for Atmospheric and Oceanic Simulation.  
amafos is developed, tested and maintained at the IT Center for Oceanography (ITCO).

What you can find on this page:

- Paper history of amafos
- Software related to amafos
- Amafos download page
- The amafos development team
- User manual for amafos
- User contribution to amafos
- Funding projects for amafos
- User team (developers, maintainers, supporters)
- The amafos project lead team and steering system (see menu above)

Example grids created with amafos

# Course Information

## Literature

Examples!

**G. Bärwolff:** *Höhere Mathematik für Naturwissenschaftler und Ingenieure* (2. Aufl.), Springer, Berlin/Heidelberg, 2009.

**R. Ansgore, H.J. Oberle, K. Rothe, T. Sonar:** *Mathematik für Ingenieure 2* (4. Aufl.), Wiley-VCH, Berlin, 2011.

**I. Gasser:** *Skriptum zur Vorlesung DGL II*, Sommersemester 2016.  
<https://www.math.uni-hamburg.de/teaching/export/tuhh/cm/d2/>

## Tables of Formulas

**K.Vetters:** *Formeln und Fakten im Grundkurs Mathematik*, Vieweg+Teubner Verlag, Wiesbaden, 2004.

## Exercises

Dr. Kai Rothe <https://www.math.uni-hamburg.de/home/rothe/>  
Dr. Hanna Peywand Kiani <https://www.math.uni-hamburg.de/home/kiani/>

Material:  
<https://1to.de/pauze>



Please prepare carefully!

# Literature

Examples!

**G. Bärwolff:** *Höhere Mathematik für Naturwissenschaftler und Ingenieure* (2. Aufl.), Springer, Berlin/Heidelberg, 2009.

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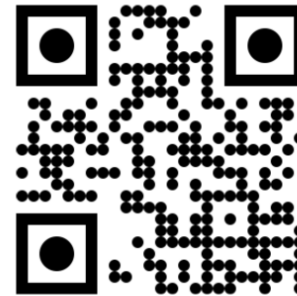
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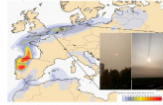


# Motivation

Transport Equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$$

Transport of Forest Fire Plumes



Transport of Volcano Ash Clouds



Eyjafjallajökull Eruption, May 2010



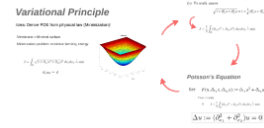
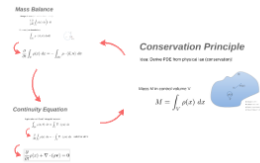
Satellite Observation      Simulation

Arctic Ozone Depletion



Oil Spill

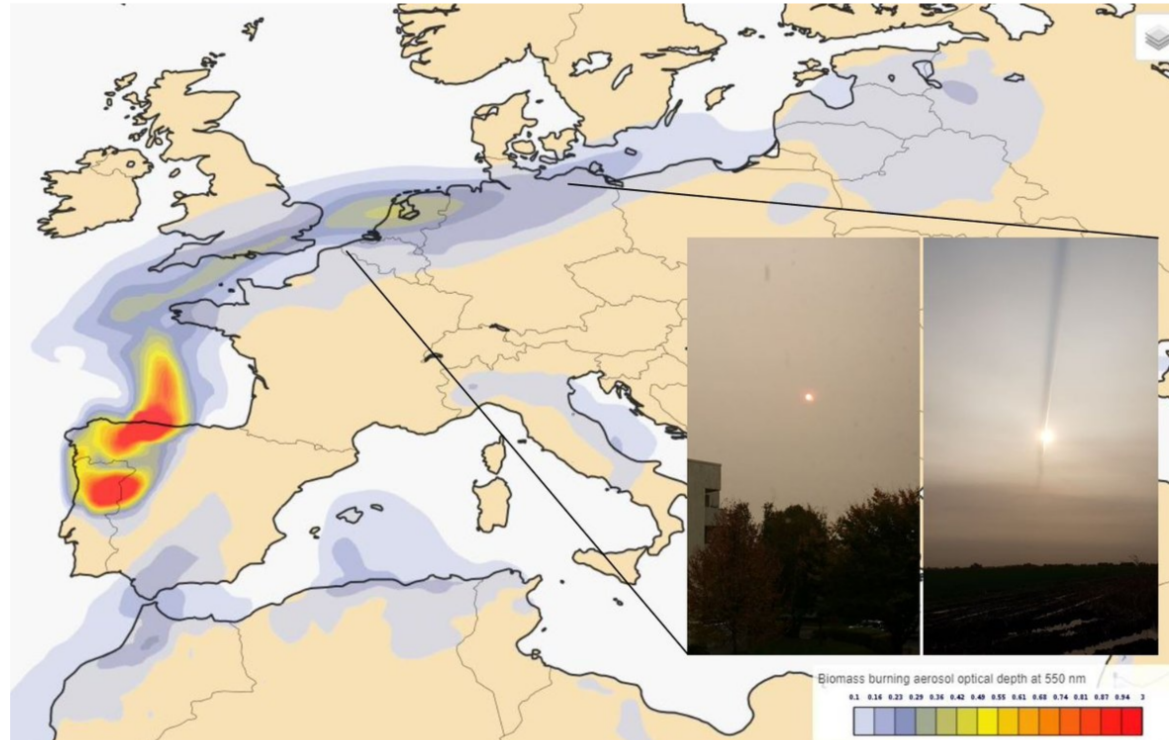
Satellite Pictures of Deepwater Horizon Disaster  
April 2010



# Transport Equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$$

# Transport of Forest Fire Plumes

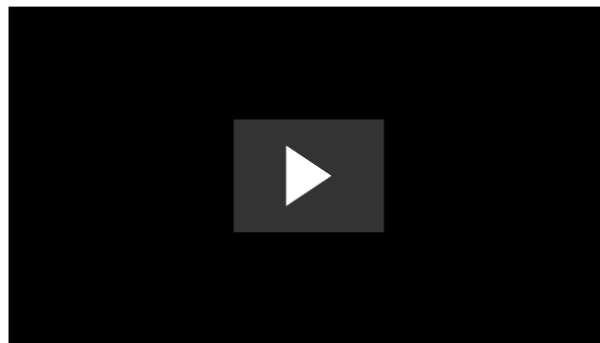


<http://www.severe-weather.eu/news/north-central-europe-under-thick-smoke-from-fires-in-north-spain-and-portugal-today-oct-17-2017/>

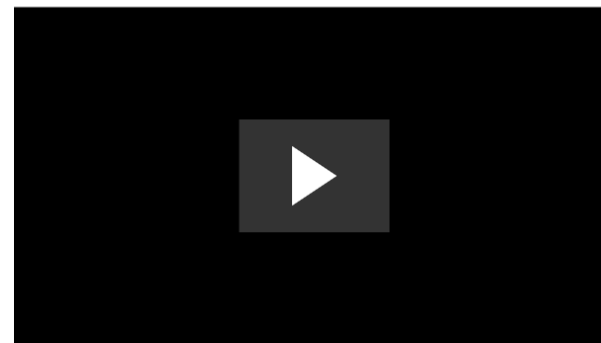
# Transport of Volcano Ash Clouds



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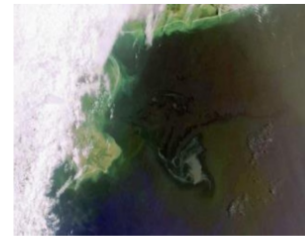
Satellite Observation



Simulation

# Oil Spill

Satellite Pictures of Deepwater Horizon Disaster  
April 2010

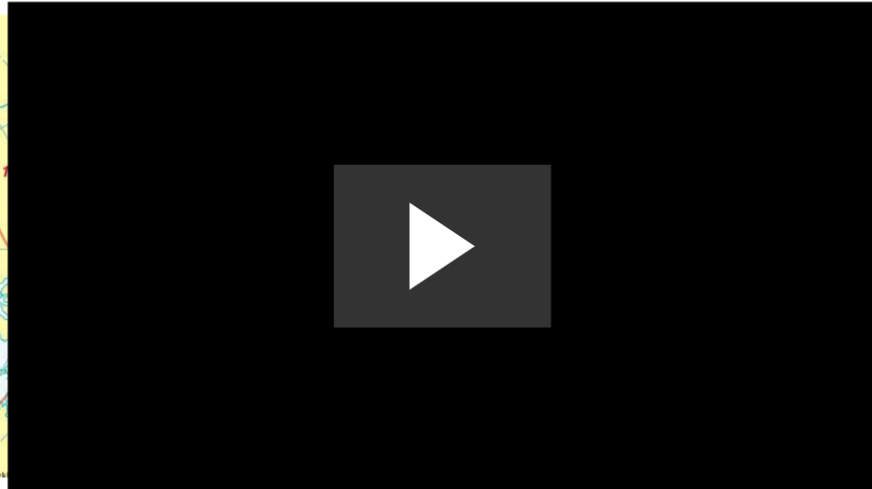


# Arctic Ozone Depletion

Given wind field in Arctic stratosphere (approx. 18 km height)



<http://www.lib.utexas.edu/maps>

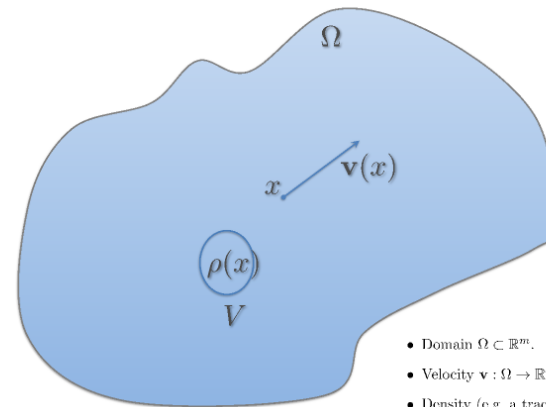


# Conservation Principle

Idea: Derive PDE from physical laws (conservation)!

Mass  $M$  in control volume  $V$

$$M = \int_V \rho(x) dx$$



- Domain  $\Omega \subset \mathbb{R}^n$ .
- Velocity  $\mathbf{v} : \Omega \rightarrow \mathbb{R}^d$ .
- Density (e.g. a tracer)  $\rho : \Omega \rightarrow \mathbb{R}$ .
- Control Volume  $V \subset \Omega$ .

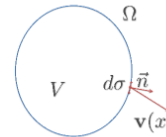
# Mass Balance

Change of mass (infinitesimal time interval):

$$\left( \frac{\partial}{\partial t} \int_V \rho(x) dx \right) \cdot dt$$

Flux rate (over boundaries)

$$\int_{\partial V} \rho \cdot \langle \vec{n}, \mathbf{v} \rangle d\sigma dt$$



$$\frac{\partial}{\partial t} \int_V \rho(x) dx = - \int_{\partial V} \rho \cdot \langle \vec{n}, \mathbf{v} \rangle d\sigma$$





# Continuity Equation

Application of Gauß' integral theorem:

$$\int_{\partial V} \rho \langle \mathbf{v}, \vec{n} \rangle d\sigma = \int_V \nabla \cdot (\rho \mathbf{v}) dx$$

$$\frac{\partial}{\partial t} \int_V \rho(x) dx = - \int_V \nabla \cdot (\rho \mathbf{v}) dx \quad \text{valid for all } V$$

$$\frac{\partial}{\partial t} \rho(x) + \nabla \cdot (\rho \mathbf{v}) = 0$$

# Variational Principle

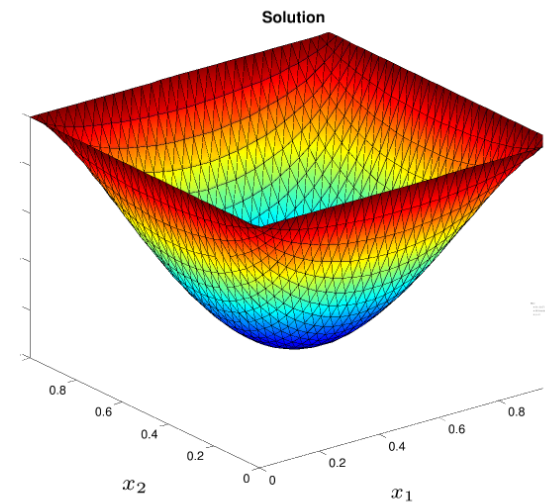
Idea: Derive PDE from physical law (Minimization)!

Membrane = Minimal surface

Minimization problem: minimize bending energy

$$J = \int_{\Omega} \sqrt{1 + \partial_{x_1} u^2 + \partial_{x_2} u^2} \, dx_1 dx_2 \stackrel{!}{=} \min$$

$$u|_{\partial\Omega} = \phi$$



# Derivation

For

$$\nabla u = (\partial_{x_1} u, \partial_{x_2} u) \ll 1,$$

(i.e.  $\nabla u$  small), assume


$$\sqrt{1 + \partial_{x_1}^2 u + \partial_{x_2}^2 u} \approx 1 + \frac{1}{2}(\partial_{x_1}^2 u + \partial_{x_2}^2 u)$$



$$J = \frac{1}{2} \int_{\Omega} (\partial_{x_1}^2 u^2 + \partial_{x_2}^2 u^2) dx_1 dx_2 \stackrel{!}{=} \min$$



# Poisson's Equation



Set:  $F(u, \partial_{x_1} u, \partial_{x_2} u) := \partial_{x_1} u^2 + \partial_{x_2} u^2$

Then it holds:

If  $J = \frac{1}{2} \int_{\Omega} (\partial_{x_1} u^2 + \partial_{x_2} u^2) dx_1 dx_2 \stackrel{!}{=} \min$

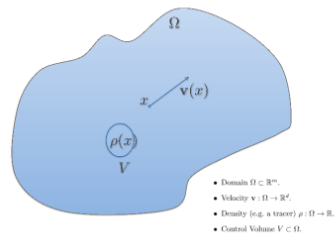
$$\Delta u := (\partial_{x_1}^2 + \partial_{x_2}^2) u = 0$$

## Conservation Principle

Idea: Derive PDE from physical law (conservation)!

Mass M in control volume V

$$M = \int_V \rho(x) dx$$



## Variational Principle

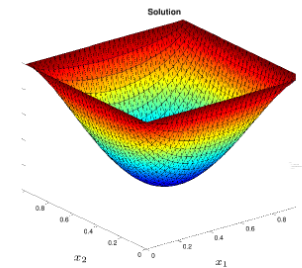
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$$u|_{\partial\Omega} = \phi$$



# Notations and Definitions

**Definition:** (Partial Differential Equation)

An equation resp. a system of equations of the form

$$F\left(x, u(x), \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \dots, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^m u}{\partial x_1^m}\right) = 0$$

for an unknown function  $u: D \rightarrow \mathbb{R}^m$ ,  $D \subset \mathbb{R}^n$ , is called **system of partial differential equations (PDE)** for the  $m$  functions  $u_1(x), \dots, u_m(x)$ .

If one of the partial derivatives occurs explicitly and is of  $p^{\text{th}}$  order  $\left(\frac{\partial^p u}{\partial x_1^p} = \frac{\partial^p u}{\partial x_1^p}\right)$ , then we call the PDE of **order  $p$** .

**Remarks:**

- In applications we often use **spatial variables**:  $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ . Often times  $n = 3$ .

- Correspondingly, we often use a **time variable**:  $t \in \mathbb{R}$ .

- Then, consider the general PDE in  $(n+1)$  variables

$$F\left(x, t, u(x, t), \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t}, \dots, \frac{\partial^2 u}{\partial x_1^2}, \dots, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 u}{\partial t^2}\right) = 0$$

- Differential operators such as

$$\nabla, \operatorname{div}, \operatorname{rot}, \Delta$$

then apply to the  $n$  spatial variables, e.g.

$$\operatorname{div} = \sum_{i=1}^n \frac{\partial}{\partial x_i}$$

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

**Definition:** (Linear/non-linear PDE)

1. A PDE is called **linear**, if  $F(x, u, \dots)$  is affine linear in variables  $u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^p u}{\partial x_1^p}$ .
2. A PDE is called **semi-linear**, if  $F(x, u, \dots)$  is affine linear in variables  $\frac{\partial u}{\partial x_1}, \dots, \frac{\partial^p u}{\partial x_1^p}$  and the coefficients depend only on  $x = (x_1, \dots, x_n)^T$ .
3. A PDE is called **quasi-linear**, if  $F(x, u, \dots)$  is affine linear in variables  $\frac{\partial u}{\partial x_1}, \dots, \frac{\partial^p u}{\partial x_1^p}$ . The coefficients may depend on  $x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^{p-1} u}{\partial x_1^{p-1}}$ .
4. Otherwise we call the PDE **non-linear**.

**Examples:**

1. Scalar linear PDE of 1<sup>st</sup> order in two variables:

$$u_1(x, y)u_x + u_2(x, y)u_y + h(x, y)u = r(x, y)$$

2. Scalar quasi-linear PDE of 1<sup>st</sup> order in two variables:

$$u_1(x, y, u)u_x + u_2(x, y, u)u_y = g(x, y, u)$$

3. Semi-linear system of PDEs of 2<sup>nd</sup> order in  $n$  variables:

$$\sum_{i,j=1}^n a_{ij}(x_1, \dots, x_n, u)u_{x_i x_j} = f(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n})$$

4. Non-linear scalar PDE of 1<sup>st</sup> order in two variables:

$$(u_x)^2 + (u_y)^2 = f(x, y, u, u_x, u_y)$$

**Definition:** (Partial Differential Equation)

An equation resp. a system of equations of the form

$$\mathbf{F} \left( \mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial \mathbf{u}}{\partial x_n}, \dots, \frac{\partial^p \mathbf{u}}{\partial^{p-1} x_1 \partial x_2}, \dots, \frac{\partial^p \mathbf{u}}{\partial^p x_n} \right) = \mathbf{0}$$

for an unknown function  $\mathbf{u} : D \rightarrow \mathbb{R}^m$ ,  $D \subset \mathbb{R}^n$ , is called **system of partial differential equations** (PDE) for the  $m$  functions  $u_1(\mathbf{x}), \dots, u_m(\mathbf{x})$ .

If one of the partial derivatives occurs explicitly and is of  $p^{\text{th}}$  order  $\left( \frac{\partial^p \mathbf{u}}{\partial^{p_1} x_1 \dots \partial^{p_n} x_n} \right)$ , then we call the **PDE of order  $p$** .

**Remark:** In applications we see typically (systems of) PDE of **first and second order**.

**Definition:** (linear/non-linear PDE)

1. A PDE is called **linear**, if  $\mathbf{F}(\mathbf{x}, \mathbf{u}, \dots)$  is affine linear in variables  $\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$ .
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The coefficients may depend on  $\mathbf{x}, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial^{p-1} \mathbf{u}}{\partial x_n^{p-1}}$ .
4. Otherwise we call the PDE **non-linear**.



## Examples:

1. Scalar linear PDE of 1<sup>st</sup> order in two variables:

$$a_1(x, y)u_x + a_2(x, y)u_y + b(x, y)u = c(x, y).$$

2. Scalar quasi-linear PDE of 1<sup>st</sup> order in two variables:

$$a_1(x, y, u)u_x + a_2(x, y, u)u_y + b(x, y, u)u = g(x, y, u).$$

3. Semi-linear system of PDEs of 2<sup>nd</sup> order in  $n$  variables:

$$\sum_{i,j=1}^n a_{ij}(x_1, \dots, x_n)u_{x_i x_j} = b(x_1, \dots, x_n, \mathbf{u}, \mathbf{u}_{x_1}, \dots, \mathbf{u}_{x_n}).$$

4. Non-linear scalar PDE of 1<sup>st</sup> order in two variables:

$$(u_x)^2 + (u_y)^2 = f(x, y, u, u_x \cdot u_y).$$

**Remarks:**

- In applications we often use **spatial variables**:  $\mathbf{x} = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$ . Oftentimes  $n = 3$ .
- Correspondingly, we often use a **time variable**:  $t \in \mathbb{R}$ .
- Then, consider the general PDE in  $(n + 1)$  variables

$$\mathbf{F} \left( \mathbf{x}, t, \mathbf{u}(\mathbf{x}, t), \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial \mathbf{u}}{\partial x_n}, \frac{\partial \mathbf{u}}{\partial t}, \dots, \frac{\partial^p \mathbf{u}}{\partial x_1^p}, \dots, \frac{\partial^p \mathbf{u}}{\partial t^p} \right) = \mathbf{0}.$$

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$$\operatorname{div} = \sum_{i=1}^n \frac{\partial}{\partial x_i}$$
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# Mathematical Modeling with PDEs

## Reynold's Transport Theorem



Osborne Reynolds (1842-1932)

**Theorem (Reynold's Transport Theorem)**  
 For an arbitrary differentiable scalar function  $f: D \rightarrow \mathbb{R}$ ,  $D \subset \mathbb{R}^3$  is holds:  

$$\frac{d}{dt} \int_{D(t)} f(x,t) dx = \int_{D(t)} \left[ \frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{v}) \right] dx, 0 \leq t \leq T$$

Remember:  $\mathbf{v} = \mathbf{v}(x,t)$

## Continuity Equation

**Physical Interpretation:**  
 • Conservation of mass in a control volume  $D(t)$   
 • Continuity equation:  $\frac{d}{dt} \int_{D(t)} \rho dx = 0$   
 • Mass flux:  $\int_{\partial D(t)} \rho \mathbf{v} \cdot \mathbf{n} dx = 0$   
 • Continuity equation:  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$   
 • Mass flux:  $\int_{\partial D(t)} \rho \mathbf{v} \cdot \mathbf{n} dx = 0$

## Diffusion Equation

**Heat Conduction in Solids:**  
 • For the heat conduction in a solid:  
 - Continuity equation  
 - Fourier's law  
 • For the heat conduction in a solid:  
 $\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (\kappa \nabla T) + Q$

**Remark:**  
 The Fourier's Law:  $\mathbf{q} = -\kappa \nabla T$   
 •  $\kappa$  is the thermal conductivity  
 •  $\mathbf{q}$  is the heat flux  
 •  $T$  is the temperature  
**Observation:**  
 These 2 physical problems lead to the same mathematical equation!



# Reynold's Transport Theorem

**Preliminary Remark:**

At time  $t = 0$  a physical constituent is assumed to cover a domain (bounded open set)  $D_0 \subset \mathbb{R}^n$ .

Constituents may be *electric charge* (electro dynamics) or *density* (fluid dynamics).

The function  $\phi(\mathbf{y}, t)$  describes the rate of change of a point  $\mathbf{y}_0 \in D_0$  in time:

$$\phi : D_0 \times [0, T] \rightarrow D_t \subset \mathbb{R}^n, \quad D_t := \{\phi(\mathbf{y}, t) : \mathbf{y} \in D_0\}.$$

The **trajectory** of  $\mathbf{y} \in D_0$  is the mapping

$$\tau : t \mapsto \phi(\mathbf{y}, t) \in D_t$$

and

$$\frac{\partial}{\partial t} \phi(\mathbf{y}, t) =: \mathbf{v}(\phi(\mathbf{y}, t), t)$$

denotes the **velocity field**  $\mathbf{v}$  of the physical constituent.



Osborne Reynolds (1842-1912)

**Theorem:** (Reynold's Transport Theorem)

For an arbitrary differentiable, scalar function  $f : D_t \times [0, T] \rightarrow \mathbb{R}$  it holds:

$$\frac{d}{dt} \int_{D_t} f(\mathbf{x}, t) \, d\mathbf{x} = \int_{D_t} \left[ \frac{\partial}{\partial t} f + \nabla \cdot (f\mathbf{v}) \right] (\mathbf{x}, t) \, d\mathbf{x}.$$

Remember:  $\text{div } \mathbf{y} = \nabla \cdot \mathbf{y}$ .

Idea of proof:

- Transform left hand side from  $D_t$  to  $D_0$ :
 
$$\int_{D_t} f(\mathbf{x}, t) \, d\mathbf{x} = \int_{D_0} f(\phi(\mathbf{y}, t), t) \, d\mathbf{y}$$
 where  $\int_{D_0} f(\phi(\mathbf{y}, t), t) \, d\mathbf{y}$  is Jacobian matrix of  $\phi(\mathbf{y}, t)$  relative to  $\mathbf{y}$ .
- Compute time derivative of right hand side:
 
$$\frac{d}{dt} \int_{D_0} f(\phi(\mathbf{y}, t), t) \, d\mathbf{y}$$
- Transform back to the time dependent domain  $D_t$ .

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**Theorem:** (Reynold's Transport Theorem)

For an arbitrary differentiable, scalar function  $f : D_t \times [0, T] \rightarrow \mathbb{R}$  it holds:

$$\frac{d}{dt} \int_{D_t} f(\mathbf{x}, t) d\mathbf{x} = \int_{D_t} \left[ \frac{\partial}{\partial t} f + \nabla \cdot (f\mathbf{v}) \right] (\mathbf{x}, t) d\mathbf{x}.$$

Remember:  $\operatorname{div} \mathbf{y} = \nabla \cdot \mathbf{y}$ .

### Idea of proof:

- Transform left hand side from  $D_t$  to  $D_0$ :

$$\int_{D_t} f(\mathbf{x}, t) d\mathbf{x} = \int_{D_0} f(\phi(\mathbf{y}, t)) J(\mathbf{y}, t) d\mathbf{y},$$

where  $J(\mathbf{y}, t) = \det(D_{\mathbf{y}}\phi(\mathbf{y}, t))$  is Jacobi matrix of  $\phi(\mathbf{y}, t)$  corresp. to  $\mathbf{y}$ .

- Compute time derivative of right hand side:

$$\frac{d}{dt} \int_{D_0} f(\phi(\mathbf{y}, t)) J(\mathbf{y}, t) d\mathbf{y}.$$

- Transform back to the time dependent domain  $D_t$ .

# Continuity Equation

## Continuity Equation:

- Let  $\rho(\mathbf{x}, t)$  be mass density of a physical constituent (e.g. fluid density).
- Assume a **conservation principle** of the form

$$\frac{d}{dt} \int_{D_t} \rho(\mathbf{x}, t) \, d\mathbf{x} = 0.$$

- According to Reynold's transport theorem it holds:

$$\int_{D_t} \left[ \frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) \right] (\mathbf{x}, t) \, d\mathbf{x} = 0.$$

- Since  $D_t \subset \mathbb{R}^n$  arbitrary subset, the PDE (**continuity equation**) holds:

$$\frac{\partial}{\partial t} \rho(\mathbf{x}, t) + \nabla \cdot (\rho \mathbf{v})(\mathbf{x}, t) = 0.$$

## Flux Function:

- Write the continuity eq. by means of a **flux function**  $\mathbf{q}(\mathbf{x}, t)$ :

$$\frac{\partial}{\partial t} \rho(\mathbf{x}, t) + \nabla \cdot \mathbf{q}(\mathbf{x}, t) = 0.$$

- Avoid *two unknowns*  $\rho$  and  $\mathbf{q}$  in *one equation* by

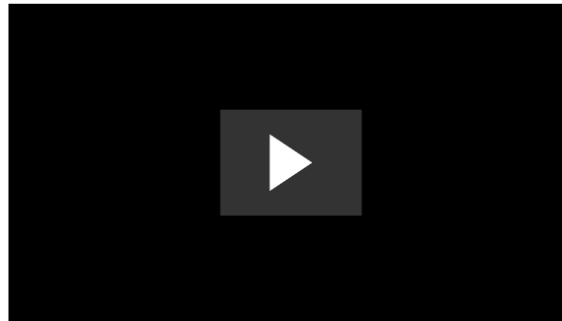
$$\mathbf{q}(\mathbf{x}, t) = \mathbf{q}(\rho(\mathbf{x}, t), \nabla \rho(\mathbf{x}, t), \dots).$$

- Example: the flux  $\mathbf{q}$  is proportional to the density  $\rho$ , i.e.

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{a} \cdot \rho(\mathbf{x}, t), \quad \mathbf{a} \in \mathbb{R}^n.$$

- Then we obtain the (**transport equation**):

$$\frac{\partial}{\partial t} \rho(\mathbf{x}, t) + \mathbf{a} \cdot \nabla \rho(\mathbf{x}, t) = 0.$$





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## Flux Function:

- Write the continuity eq. by means of a **flux function**  $\mathbf{q}(\mathbf{x}, t)$ :

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# Diffusion Equation

## Heat Conduction or Diffusion:

- Let the function  $\rho(\mathbf{x}, t)$  denote a:
  - chemical concentration
  - temperature
  - electro-static potential
- Let the flux  $\mathbf{q}$  be proportional to the gradient of  $\rho$  (with reversed sign):

$$\mathbf{q}(\mathbf{x}, t) = -a\nabla\rho(\mathbf{x}, t), \quad 0 < a \in \mathbb{R}.$$

- Then:

$$\begin{aligned} \frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot (-a\nabla\rho(\mathbf{x}, t)) &= 0 \\ \Rightarrow \frac{\partial}{\partial t}\rho(\mathbf{x}, t) &= a\Delta\rho(\mathbf{x}, t). \end{aligned}$$

- With  $a := 1$  we obtain the **heat equation** (or diffusion equation):

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) = \Delta\rho(\mathbf{x}, t).$$

## Remark:

The flux relation

$$\mathbf{q}(\mathbf{x}, t) = -a\nabla\rho(\mathbf{x}, t)$$

is called

- **Fick's law** of diffusion,
- **Fourier's law** of heat conduction, or
- **Ohm's law** of electric charge.

## Observation:

Three different physical problems lead to the same mathematical equation!

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# Motivation

Transport Equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$$



# Notations and Definitions

Define:  $\mathbf{v} = (v_x, v_y, v_z)$  is the velocity vector field.  $\rho = \rho(x, y, z, t)$  is the density.  $\mathbf{v} \cdot \nabla \rho = v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z}$  is the material derivative.

**Notes:**  
1. The velocity vector field  $\mathbf{v}$  is a vector field.  
2. The density  $\rho$  is a scalar field.  
3. The material derivative  $\mathbf{v} \cdot \nabla \rho$  is a scalar field.  
4. The material derivative  $\mathbf{v} \cdot \nabla \rho$  is the rate of change of  $\rho$  following a particle moving with velocity  $\mathbf{v}$ .

- Exercises:**
1. Let  $\mathbf{v} = (v_x, v_y, v_z)$  be a velocity vector field. Compute  $\mathbf{v} \cdot \nabla \rho$  for  $\rho = x^2 + y^2 + z^2$ .
  2. Let  $\mathbf{v} = (v_x, v_y, v_z)$  be a velocity vector field. Compute  $\mathbf{v} \cdot \nabla \rho$  for  $\rho = x^2 + y^2 + z^2$ .
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# Mathematical Modeling with PDEs

Reynold's Transport Theorem



Continuity Equation

Diffusion Equation



# Differential Equations II



# Course Information

**Course:** Differential Equations II  
**Prerequisites:** Differential Equations I, Linear Algebra  
**Topics:** Second-order linear differential equations, Sturm-Liouville theory, Bessel functions, Legendre polynomials, Hypergeometric functions, Asymptotic methods, Phase plane analysis, Nonlinear differential equations, Stability theory, Perturbation methods, Integral equations, Variational calculus, Boundary value problems, Eigenvalue problems, Green's functions, Fourier series, Fourier transforms, Laplace transforms, Z-transforms, Discrete-time systems, Control systems, Signal processing, Numerical methods, Applications in physics, engineering, and biology.

## Exercises



# Your Professor

**Coordinates:**



**Address:**

**Research Interests:**

