Differential Equations II

Jörn Behrens

Introduction

Your Professor

Coordinates:

Prof. Dr. Jörn Behrens
Uni Hamburg/ Dept. Mathematics/ CEN
Grindelberg 5/ Bundesstraße 55
Tel. (040) 42838 7734
mail joern.behrens@uni-hamburg.de

Office hours after appointment by email

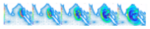
Background

Short CV

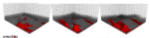
since 2009 Prof. @ Uni Hamburg, Dept. Mathematics
2006-2009 Lead Tsunami Group @ AWI, Adjunct @ Uni Bremen
2005 Habilitation (Mathematics) @ TUM
2003-2004 Visiting Scientist @ NCAR, Boulder, CO, USA
1998-2006 Assistant Prof (Wiss. Rat) @ TUM, Scient. Computing
1996-1998 Post-Doc @ AWI
1991-1996 Dr. rer. nat. (Mathematics) @ AWI/Uni Bremen
1991 Diploma Mathematics @ Uni Bonn

Research Interests

Adaptive Tsunami Modeling



Adaptive Atmospheric Modeling



Mesh Generation

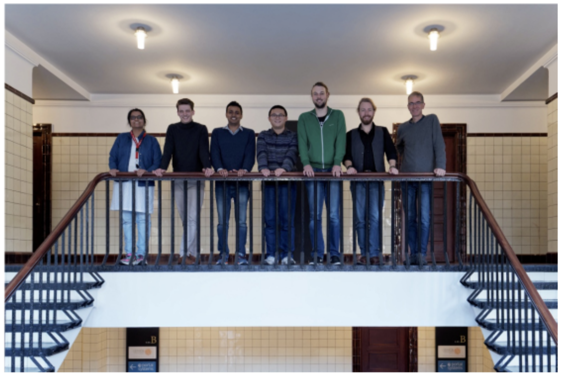
Multi-Scale Simulation

Coordinates:

Prof. Dr. Jörn Behrens
Uni Hamburg/ Dept. Mathematics/ CEN
Grindelberg 5/ Bundesstraße 55
Tel. (040) 42838 7734
mail joern.behrens@uni-hamburg.de

Office hours after appointment by email

Background



Short CV

since 2009 Prof. @ Uni Hamburg, Dept. Mathematics
2006-2009 Lead Tsunami Group @ AWI, Adjunct @ Uni Bremen
2005 Habilitation (Mathematics) @ TUM
2003-2004 Visiting Scientist @ NCAR, Boulder, CO, USA
1998-2006 Assistant Prof (Wiss. Rat) @ TUM, Scient. Computing
1996-1998 Post-Doc @ AWI
1991-1996 Dr. rer. nat. (Mathematics) @ AWI/Uni Bremen
1991 Diploma Mathematics @ Uni Bonn

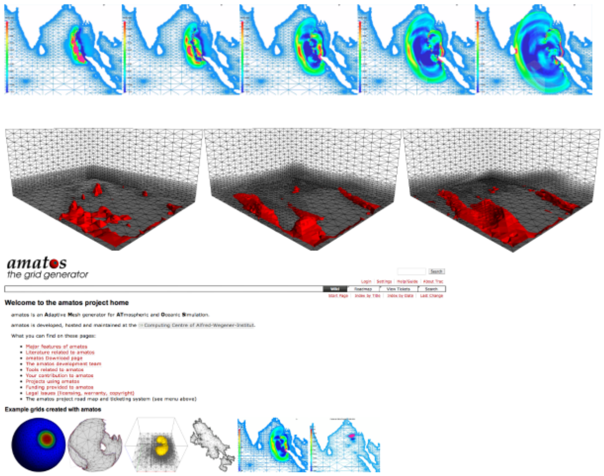
Research Interests

Adaptive Tsunami Modeling

Adaptive Atmospheric Modeling

Mesh Generation

Multi-Scale Simulation



Course Information

Literature

Examples!

G. Bärwolff: *Höhere Mathematik für Naturwissenschaftler und Ingenieure* (2. Aufl.), Springer, Berlin/Heidelberg, 2009.

R. Ansorge, H.J. Oberle, K. Rothe, T. Sonar: *Mathematik für Ingenieure 2* (4. Aufl.), Wiley-VCH, Berlin, 2011.

I. Gasser: *Skriptum zur Vorlesung DGL II*, Sommersemester 2016.
https://www.math.uni-hamburg.de/teaching/export/tuhh/cm/d2/

Tables of Formulas

K.Vetters: *Formeln und Fakten im Grundkurs Mathematik*, Vieweg+Teubner Verlag, Wiesbaden, 2004.

Exercises

Dr. Kai Rothe https://www.math.uni-hamburg.de/home/rothe/
Dr. Hanna Peywand Kiani https://www.math.uni-hamburg.de/home/kiani/

Material:
https://t1p.de/pauzc

Please prepare carefully!

Literature

Examples!

G. Bärwolff: *Höhere Mathematik für Naturwissenschaftler und Ingenieure* (2. Aufl.), Springer, Berlin/Heidelberg, 2009.

R. Ansorge, H.J. Oberle, K. Rothe, T. Sonar: *Mathematik für Ingenieure 2* (4. Aufl.), Wiley-VCH, Berlin, 2011.

I. Gasser: *Skriptum zur Vorlesung DGL II*, Sommersemester 2016.
https://www.math.uni-hamburg.de/teaching/export/tuhh/cm/d2/

Tables of Formulas

K.Vetters: *Formeln und Fakten im Grundkurs Mathematik*, Vieweg+Teubner Verlag, Wiesbaden, 2004.

Exercises

Dr. Kai Rothe https://www.math.uni-hamburg.de/home/rothe/

Dr. Hanna Peywand Kiani https://www.math.uni-hamburg.de/home/kiani/

Material:

https://t1p.de/pauzc

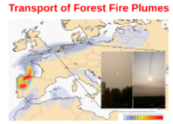
Please prepare carefully!

Motivation

Transport Equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$$

Transport of Forest Fire Plumes



Transport of Volcano Ash Clouds

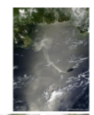
Eyjafjallajokull Eruption, May 2010

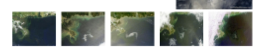
Satellite Observation Simulation

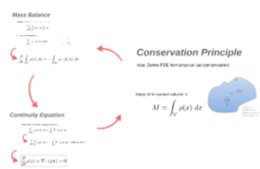
Arctic Ozone Depletion

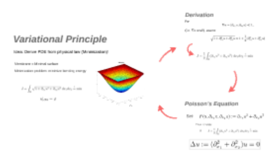
Oil Spill

Satellite Pictures of Deepwater Horizon Disaster
April 2010





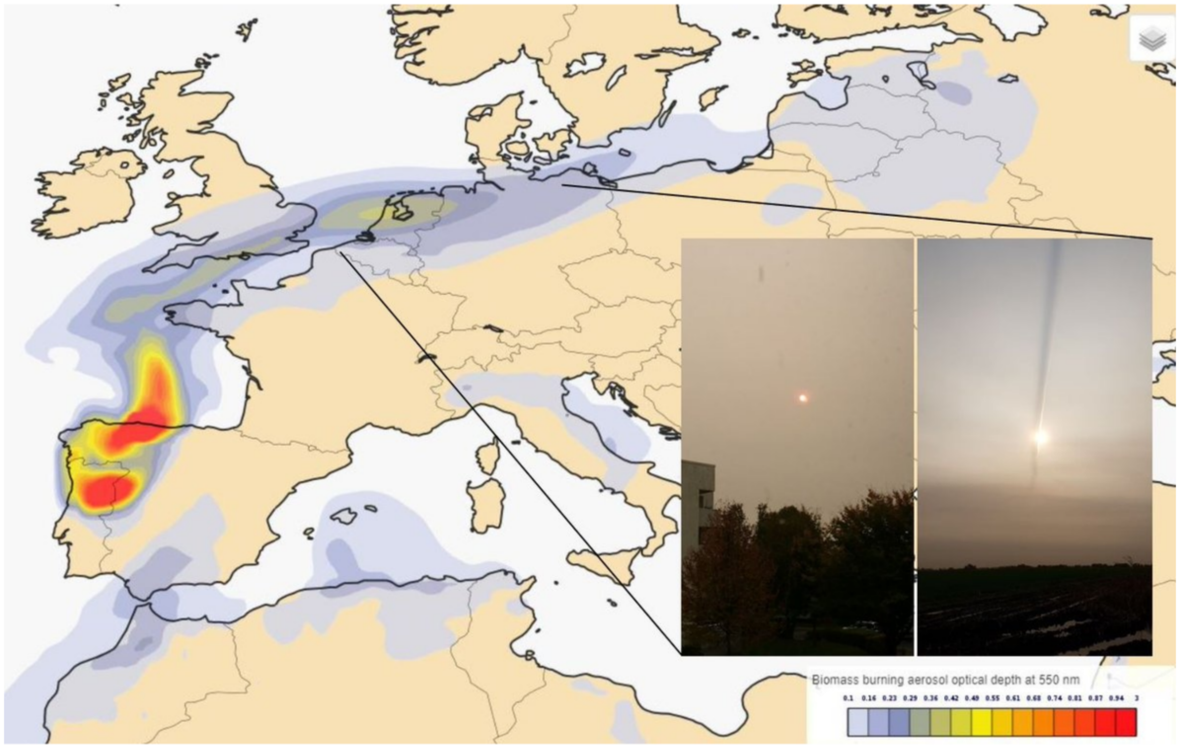




Transport Equation:

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$$

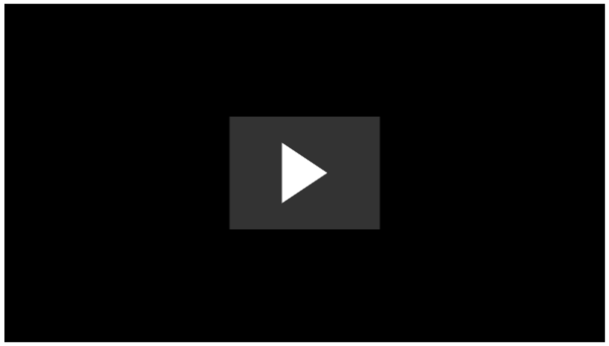
Transport of Forest Fire Plumes



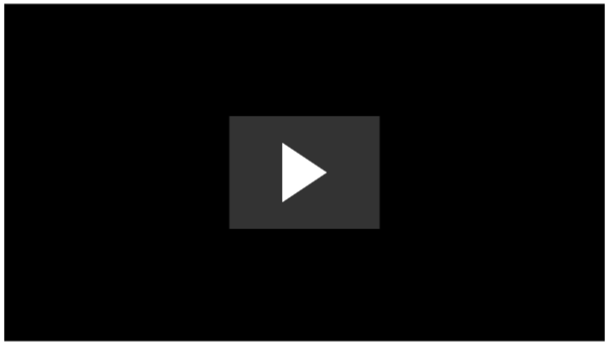
http ://www.severe-weather.eu/news/north-central-europe-under-thick-smoke-from-fires-in-north-spain-and-portugal-today-oct-17-2017/

Transport of Volcano Ash Clouds

Eyjafjallajokull Eruption, May 2010



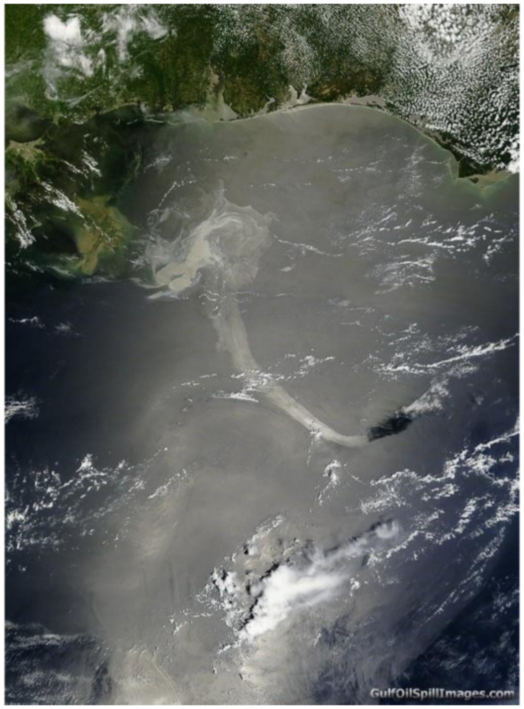
Satellite Observation

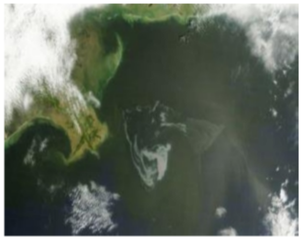


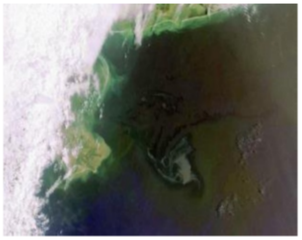
Simulation

Oil Spill

Satellite Pictures of Deepwater Horizon Disaster
April 2010



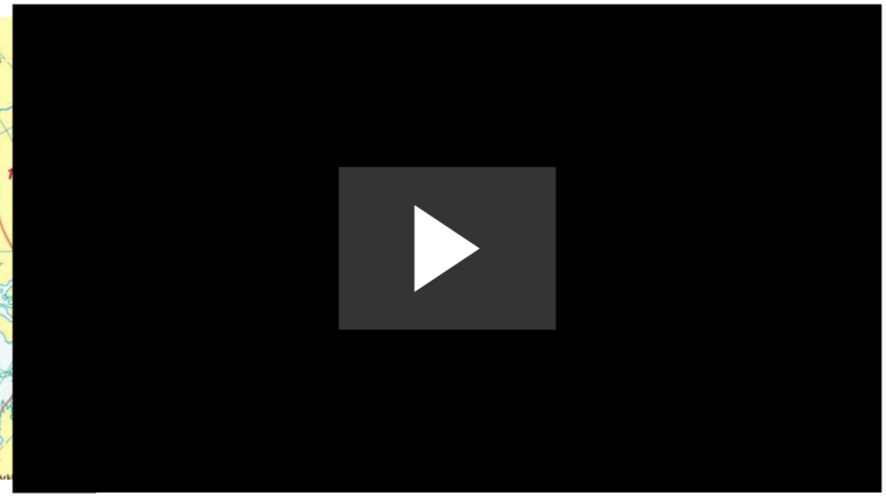




Arctic Ozone Depletion

Given wind field in Arctic stratosphere (approx. 18 km height)

http://www.lib.utexas.edu/maps



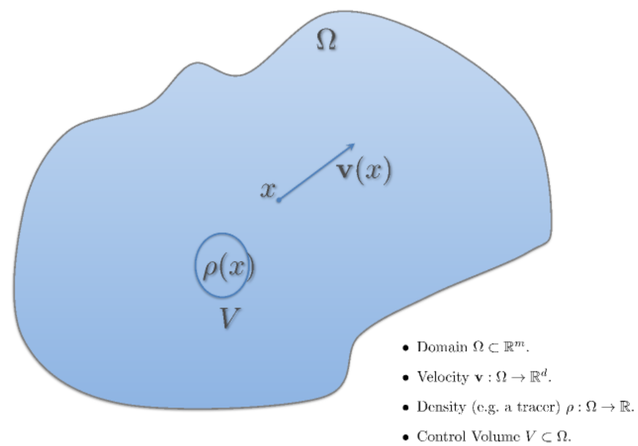
Transport of ozone depleted air parcel

Conservation Principle

Idea: Derive PDE from physical lae (conservation)!

Mass M in control volume V

$$M = \int_V \rho(x)\ dx$$



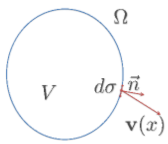
Mass Balance

Change of mass (infinitesimal time interval):

$$\left(\frac{\partial}{\partial t} \int_v \rho(x)\ dx \right) \cdot dt$$

Flux rate (over boundaries)

$$\int_{\partial V} \rho \cdot \langle \vec{n}, \mathbf{v} \rangle\ d\sigma dt$$



$$\frac{\partial}{\partial t} \int_V \rho(x)\ dx = - \int_{\partial V} \rho \cdot \langle \vec{n}, \mathbf{v} \rangle\ d\sigma$$

Continuity Equation

Application of Gauß' integral theorem:

$$\int_{\partial V} \rho \langle \mathbf{v}, \vec{n} \rangle \; d\sigma = \int_V \nabla \cdot (\rho \mathbf{v}) \; dx$$

$$\frac{\partial}{\partial t} \int_V \rho(x) \; dx = - \int_V \nabla \cdot (\rho \mathbf{v}) \; dx$$ valid for all V

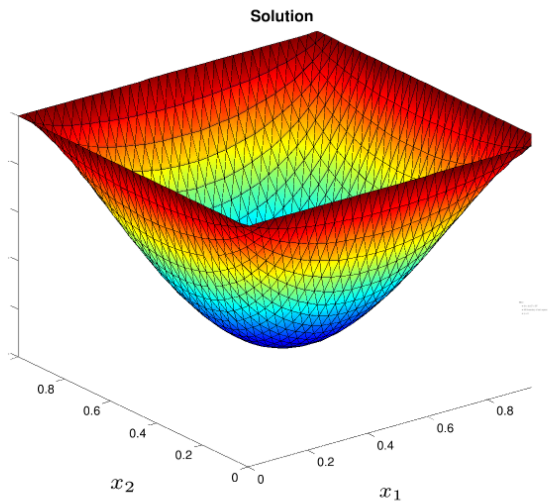
$$\frac{\partial}{\partial t} \rho(x) + \nabla \cdot (\rho \mathbf{v}) = 0$$

Variational Principle

Idea: Derive PDE from physical law (Minimization)!

Membrane = Minimal surface

Minimization problem: minimize bending energy



$$J = \int_\Omega \sqrt{1 + \partial_{x_1} u^2 + \partial_{x_2} u^2}\, dx_1 dx_2 \stackrel{!}{=} \min$$

$$u|_{\partial\Omega} = \phi$$

Derivation

For

$$\nabla u = (\partial_{x_1} u, \partial_{x_2} u) \ll 1,$$

(i.e. ∇u small), assume

$$\sqrt{1 + \partial_{x_1}^2 u + \partial_{x_2}^2 u} \approx 1 + \frac{1}{2}(\partial_{x_1}^2 u + \partial_{x_2}^2 u)$$

$$J = \frac{1}{2} \int_\Omega (\partial_{x_1} u^2 + \partial_{x_2} u^2)\ dx_1 dx_2 \stackrel{!}{=} \min$$

Poisson's Equation

Set: $F(u, \partial_{x_1} u, \partial_{x_2} u) := \partial_{x_1} u^2 + \partial_{x_2} u^2$

Then it holds:

If $J = \frac{1}{2} \int_{\Omega} (\partial_{x_1} u^2 + \partial_{x_2} u^2) \; dx_1 dx_2 \overset{!}{=} \min$

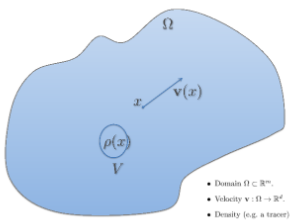
$$\Delta u := (\partial_{x_1}^2 + \partial_{x_2}^2) u = 0$$

Conservation Principle

Idea: Derive PDE from physical lae (conservation)!

Mass M in control volume V

$$M = \int_V \rho(x)\ dx$$



- Domain $\Omega \subset \mathbb{R}^m$.
- Velocity $\mathbf{v} : \Omega \to \mathbb{R}^d$.
- Density (e.g. a tracer) $\rho : \Omega \to \mathbb{R}$.
- Control Volume $V \subset \Omega$.

Variational Principle

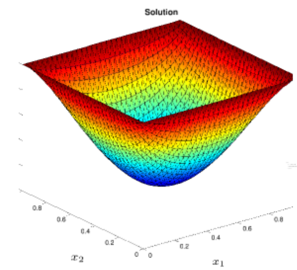
Idea: Derive PDE from physical law (Minimization)!

Membrane = Minimal surface

Minimization problem: minimize bending energy

$$J = \int_\Omega \sqrt{1 + \partial_{x_1} u^2 + \partial_{x_2} u^2}\ dx_1 dx_2 \stackrel{!}{=} \min$$

$$u|_{\partial\Omega} = \phi$$



Notations and Definitions

Definition: (Partial Differential Equation)
An equation resp. a system of equations of the form

$$\mathbf{F}\left(\mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x_1}, \ldots, \frac{\partial \mathbf{u}}{\partial x_n}, \ldots, \frac{\partial^p \mathbf{u}}{\partial^p x_1}, \frac{\partial^p \mathbf{u}}{\partial^{p-1} x_1 \partial x_2}, \ldots, \frac{\partial^p \mathbf{u}}{\partial^p x_n}\right) = \mathbf{0}$$

for an unknown function $\mathbf{u} : D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$, is called system of partial differential equations (PDE) for the m functions $u_1(\mathbf{x}), \ldots, u_m(\mathbf{x})$.

If one of the partial derivatives occurs explicitly and is of p^{th} order $\left(\frac{\partial^p \mathbf{u}}{\partial^{p_1} x_1 \cdots \partial^{p_n} x_n}\right)$, then we call the PDE of order p.

Remarks:

- In applications we often use spatial variables: $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$. Oftentimes $n = 3$.
- Correspondingly, we often use a time variable: $t \in \mathbb{R}$.
- Then, consider the general PDE in $(n+1)$ variables

$$\mathbf{F}\left(\mathbf{x}, t, \mathbf{u}(\mathbf{x}, t), \frac{\partial \mathbf{u}}{\partial x_1}, \ldots, \frac{\partial \mathbf{u}}{\partial x_n}, \frac{\partial \mathbf{u}}{\partial t}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_1^p}, \ldots, \frac{\partial^p \mathbf{u}}{\partial t^p}\right) = \mathbf{0}.$$

- Differential operators such as

$$\nabla, \ \mathrm{div}, \ \mathrm{rot}, \Delta$$

then apply to the n spatial variables, e.g.

$$\mathrm{div} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$$
$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

Definition: (linear/non-linear PDE)

1. A PDE is called linear, if $\mathbf{F}(\mathbf{x}, \mathbf{u}, \ldots)$ is affine linear in variables $\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$.
2. A PDE is called semi-linear, if $\mathbf{F}(\mathbf{x}, \mathbf{u}, \ldots)$ is affine linear in variables $\frac{\partial^p \mathbf{u}}{\partial x_1^p}, \frac{\partial^p \mathbf{u}}{\partial x_1^{p-1} \partial x_2}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$ and the coefficients depend only on $\mathbf{x} = (x_1, \ldots, x_n)^\top$.
3. A PDE is called quasi-linear, if $\mathbf{F}(\mathbf{x}, \mathbf{u}, \ldots)$ is affine linear in variables $\frac{\partial^p \mathbf{u}}{\partial x_1^p}, \frac{\partial^p \mathbf{u}}{\partial x_1^{p-1} \partial x_2}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$. The coefficients may depend on $\mathbf{x}, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \ldots, \frac{\partial^{p-1} \mathbf{u}}{\partial x_n^{p-1}}$.
4. Otherwise we call the PDE non-linear.

Examples:

1. Scalar linear PDE of 1st order in two variables:

$$a_1(x,y)u_x + a_2(x,y)u_y + b(x,y)u = c(x,y).$$

2. Scalar quasi-linear PDE of 1st order in two variables:

$$a_1(x,y,u)u_x + a_2(x,y,u)u_y + = g(x,y,u).$$

3. Semi-linear system of PDEs of 2nd order in n variables:

$$\sum_{i,j=1}^{n} a_{ij}(x_1, \ldots, x_n) u_{x_i x_j} = b(x_1, \ldots, x_n, \mathbf{u}, \mathbf{u}_{x_1}, \ldots, \mathbf{u}_{x_n}).$$

4. Non-linear scalar PDE of 1st order in two variables:

$$(u_x)^2 + (u_y)^2 = f(x, y, u, u_x \cdot u_y).$$

Definition: (Partial Differential Equation)
An equation resp. a system of equations of the form

$$\mathbf{F}\left(\mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial \mathbf{u}}{\partial x_n}, \dots, \frac{\partial^p \mathbf{u}}{\partial^p x_1}, \frac{\partial^p \mathbf{u}}{\partial^{p-1} x_1 \partial x_2}, \dots, \frac{\partial^p \mathbf{u}}{\partial^p x_n}\right) = \mathbf{0}$$

for an unknown function $\mathbf{u} : D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$, is called system of partial differential equations (PDE) for the m functions $u_1(\mathbf{x}), \dots, u_m(\mathbf{x})$.

If one of the partial derivatives occurs explicitly and is of p^{th} order ($\frac{\partial^p \mathbf{u}}{\partial^{p_1} x_1 \cdots \partial^{p_n} x_n}$), then we call the PDE of order p.

Remark: In applications we see typically
(systems of) PDE of first and second order.

Definition: (linear/non-linear PDE)

1. A PDE is called linear, if $\mathbf{F}(\mathbf{x}, \mathbf{u}, \ldots)$ is affine linear in variables $\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$.
2. A PDE is called semi-linear, if $\mathbf{F}(\mathbf{x}, \mathbf{u}, \ldots)$ is affine linear in variables $\frac{\partial^p \mathbf{u}}{\partial x_1^p}, \frac{\partial^p \mathbf{u}}{\partial x_1^{p-1} \partial x_2}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$ and the coefficients depend only on $\mathbf{x} = (x_1, \ldots, x_n)^\top$.
3. A PDE is called quasi-linear, if $\mathbf{F}(\mathbf{x}, \mathbf{u}, \ldots)$ is affine linear in variables $\frac{\partial^p \mathbf{u}}{\partial x_1^p}, \frac{\partial^p \mathbf{u}}{\partial x_1^{p-1} \partial x_2}, \ldots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$. The coefficients may depend on $\mathbf{x}, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \ldots, \frac{\partial^{p-1} \mathbf{u}}{\partial x_n^{p-1}}$.
4. Otherwise we call the PDE non-linear.

Examples:

1. Scalar linear PDE of 1st order in two variables:

$$a_1(x,y)u_x + a_2(x,y)u_y + b(x,y)u = c(x,y).$$

2. Scalar quasi-linear PDE of 1st order in two variables:

$$a_1(x,y,u)u_x + a_2(x,y,u)u_y + = g(x,y,u).$$

3. Semi-linear system of PDEs of 2nd order in n variables:

$$\sum_{i,j=1}^{n} a_{ij}(x_1,\ldots,x_n)u_{x_i x_j} = b(x_1,\ldots,x_n,\mathbf{u},\mathbf{u}_{x_1},\ldots,\mathbf{u}_{x_n}).$$

4. Non-linear scalar PDE of 1st order in two variables:

$$(u_x)^2 + (u_y)^2 = f(x,y,u,u_x \cdot u_y).$$

Remarks:

- In applications we often use spatial variables: $\mathbf{x} = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$. Oftentimes $n = 3$.
- Correspondingly, we often use a time variable: $t \in \mathbb{R}$.
- Then, consider the general PDE in $(n+1)$ variables

$$\mathbf{F}\left(\mathbf{x}, t, \mathbf{u}(\mathbf{x}, t), \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial \mathbf{u}}{\partial x_n}, \frac{\partial \mathbf{u}}{\partial t}, \dots, \frac{\partial^p \mathbf{u}}{\partial x_1^p}, \dots, \frac{\partial^p \mathbf{u}}{\partial t^p}\right) = \mathbf{0}.$$

- Differential operators such as

$$\nabla, \ \mathrm{div}, \ \mathrm{rot}, \Delta$$

  then apply to the n spatial variables, e.g.

$$\begin{aligned} \mathrm{div} &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \\ \Delta &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \end{aligned}$$

Mathematical Modeling with PDEs

Reynold's Transport Theorem

Theorem: (Reynold's Transport Theorem)
For an arbitrary differeniable, scalar function $f : D_t \times [0,T] \to \mathbb{R}$ it holds:

$$\frac{d}{dt}\int_{D_t} f(\mathbf{x},t)\,d\mathbf{x} = \int_{D_t}\left[\frac{\partial}{\partial t}f + \nabla\cdot(f\mathbf{v})\right](\mathbf{x},t)\,d\mathbf{x}.$$

Remember: $\operatorname{div} \mathbf{y} = \nabla \cdot \mathbf{y}$.

Osborne Reynolds (1842-1912)

Continuity Equation

Diffusion Equation

Reynold's Transport Theorem

Preliminary Remark:
At time $t = 0$ a physical constituent is assumed to cover a domain (bounded open set) $D_0 \subset \mathbb{R}^n$.
Constituents may be *electric charge* (electro dynamics) or *density* (fluid dynamics).
The function $\phi(\mathbf{y}, t)$ describes the rate of change of a point $\mathbf{y}_0 \in D_0$ in time:

$$\phi : D_0 \times [0, T] \to D_t \subset \mathbb{R}^n, \quad D_t := \{\phi(\mathbf{y}, t) : \mathbf{y} \in D_0\}.$$

The trajectory of $\mathbf{y} \in D_0$ is the mapping

$$\tau : t \mapsto \phi(\mathbf{y}, t) \in D_t$$

and

$$\frac{\partial}{\partial t}\phi(\mathbf{y}, t) =: \mathbf{v}(\phi(\mathbf{y}, t), t)$$

denotes the velocity field $\mathbf{v}$ of the physical constituent.

https: //upload.wikimedia.org/wikipedia/commons/d/df/Osborne_Reynolds.jpg

Osborne Reynolds (1842-1912)

Theorem: (Reynold's Transport Theorem)
For an arbitrary differeniable, scalar function $f : D_t \times [0, T] \to \mathbb{R}$ it holds:

$$\frac{d}{dt}\int_{D_t} f(\mathbf{x}, t)\; d\mathbf{x} = \int_{D_t} \left[\frac{\partial}{\partial t} f + \nabla \cdot (f\mathbf{v})\right](\mathbf{x}, t)\; d\mathbf{x}.$$

Remember: $\operatorname{div} \mathbf{y} = \nabla \cdot \mathbf{y}$.

Idea of proof:

- Transform left hand side from D_t to D_0:

$$\int_{D_t} f(\mathbf{x}, t)\; d\mathbf{x} = \int_{D_0} f(\phi(\mathbf{y}t), t) J(\mathbf{y}, t)\; d\mathbf{y},$$

  where $J(y, t) = \det(D_y \phi(y, t))$ is Jacobi matrix of $\phi(y, t)$ corresp. to y.
- Compute time derivative of right hand side:

$$\frac{d}{dt}\int_{D_0} f(\phi(\mathbf{y}t), t) J(\mathbf{y}, t)\; d\mathbf{y}.$$

- Transform back to the time dependent domain D_t.

Preliminary Remark:
At time $t = 0$ a physical constituent is assumed to cover a domain (bounded open set) $D_0 \subset \mathbb{R}^n$.
Constituents may be *electric charge* (electro dynamics) or *density* (fluid dynamics).
The function $\phi(\mathbf{y}, t)$ describes the rate of change of a point $\mathbf{y}_0 \in D_0$ in time:

$$\phi : D_0 \times [0, T] \to D_t \subset \mathbb{R}^n, \quad D_t := \{\phi(\mathbf{y}, t) : \mathbf{y} \in D_0\}.$$

The trajectory of $\mathbf{y} \in D_0$ is the mapping

$$\tau : t \mapsto \phi(\mathbf{y}, t) \in D_t$$

and

$$\frac{\partial}{\partial t}\phi(\mathbf{y}, t) =: \mathbf{v}(\phi(\mathbf{y}, t), t)$$

denotes the velocity field $\mathbf{v}$ of the physical constituent.

Theorem: (Reynold's Transport Theorem)
For an arbitrary differeniable, scalar function $f : D_t \times [0,T] \to \mathbb{R}$ it holds:

$$\frac{d}{dt}\int_{D_t} f(\mathbf{x},t)\ d\mathbf{x} = \int_{D_t}\left[\frac{\partial}{\partial t}f + \nabla\cdot(f\mathbf{v})\right](\mathbf{x},t)\ d\mathbf{x}.$$

Remember: $\operatorname{div}\,\mathbf{y} = \nabla\cdot\mathbf{y}$.

Idea of proof:

- Transform left hand side from D_t to D_0:

$$\int_{D_t} f(\mathbf{x},t)\ d\mathbf{x} = \int_{D_0} f(\phi(\mathbf{y}t),t)J(\mathbf{y},t)\ d\mathbf{y},$$

  where $J(\mathbf{y},t) = \det(D_{\mathbf{y}}\phi(\mathbf{y},t))$ is Jacobi matrix of $\phi(\mathbf{y},t)$ corresp. to $\mathbf{y}$.

- Compute time derivative of right hand side:

$$\frac{d}{dt}\int_{D_0} f(\phi(\mathbf{y}t),t)J(\mathbf{y},t)\ d\mathbf{y}.$$

- Transform back to the time dependent domain D_t.

Continuity Equation

Continuity Equation:

- Let $\rho(\mathbf{x},t)$ be mass density of a physical constituent (e.g. fluid density).
- Assume a conservation principle of the form

$$\frac{d}{dt}\int_{D_t}\rho(\mathbf{x},t)\ d\mathbf{x}=0.$$

- According to Reynold's transport theorem it holds:

$$\int_{D_t}\left[\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\mathbf{v})\right](\mathbf{x},t)\ d\mathbf{x}=0.$$

- Since $D_t\subset\mathbb{R}^n$ arbitrary subset, the PDE (continuity equation) holds:

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t)+\nabla\cdot(\rho\mathbf{v})(\mathbf{x},t)=0.$$

Flux Function:

- Write the continuity eq. by means of a flux function $\mathbf{q}(\mathbf{x},t)$:

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t)+\nabla\cdot\mathbf{q}(\mathbf{x},t)=0.$$

- Avoid *two* unknowns ρ and $\mathbf{q}$ in *one* equation by

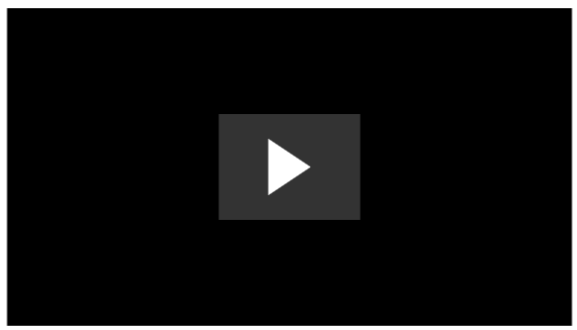
$$\mathbf{q}(\mathbf{x},t)=\mathbf{q}(\rho(\mathbf{x},t),\nabla\rho(\mathbf{x},t),\ldots).$$

- Example: the flux $\mathbf{q}$ is proportional to the density ρ, i.e.

$$\mathbf{q}(\mathbf{x},t)=\mathbf{a}\cdot\rho(\mathbf{x},t),\quad \mathbf{a}\in\mathbb{R}^n.$$

- Then we obtain the (transport equation):

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t)+\mathbf{a}\cdot\nabla\rho(\mathbf{x},t)=0.$$



Continuity Equation:

- Let $\rho(\mathbf{x}, t)$ be mass density of a physical constituent (e.g. fluid density).
- Assume a conservation principle of the form

$$\frac{d}{dt}\int_{D_t} \rho(\mathbf{x}, t)\ d\mathbf{x} = 0.$$

- According to Reynold's transport theorem it holds:

$$\int_{D_t} \left[\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{v})\right](\mathbf{x}, t)\ d\mathbf{x} = 0.$$

- Since $D_t \subset \mathbb{R}^n$ arbitrary subset, the PDE (continuity equation) holds:

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot (\rho \mathbf{v})(\mathbf{x}, t) = 0.$$

Flux Function:

- Write the continuity eq. by means of a flux function $\mathbf{q}(\mathbf{x}, t)$:

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot \mathbf{q}(\mathbf{x}, t) = 0.$$

- Avoid *two* unknowns ρ and $\mathbf{q}$ in *one* equation by

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{q}(\rho(\mathbf{x}, t), \nabla\rho(\mathbf{x}, t), \ldots).$$

- Example: the flux $\mathbf{q}$ is proportional to the density ρ, i.e.

$$\mathbf{q}(\mathbf{x}, t) = \mathbf{a} \cdot \rho(\mathbf{x}, t), \quad \mathbf{a} \in \mathbb{R}^n.$$

- Then we obtain the (transport equation):

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \mathbf{a} \cdot \nabla\rho(\mathbf{x}, t) = 0.$$

Diffusion Equation

Heat Conduction or Diffusion:

- Let the function $\rho(\mathbf{x}, t)$ denote a:
  - chemical concentration
  - temperature
  - electro-static potential
- Let the flux $\mathbf{q}$ be proportional to the gradient of ρ (with reversed sign):

$$\mathbf{q}(\mathbf{x}, t) = -a\nabla\rho(\mathbf{x}, t), \quad 0 < a \in \mathbb{R}.$$

- Then:

$$\begin{aligned} \frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot (-a\nabla\rho(\mathbf{x}, t)) &= 0 \\ \Rightarrow \quad \frac{\partial}{\partial t}\rho(\mathbf{x}, t) &= a\Delta\rho(\mathbf{x}, t). \end{aligned}$$

- With $a := 1$ we obtain the heat equation (or diffusion equation):

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) = \Delta\rho(\mathbf{x}, t).$$

Remark:
The flux relation

$$\mathbf{q}(\mathbf{x}, t) = -a\nabla\rho(\mathbf{x}, t)$$

is called

- Fick's law of diffusion,
- Fourier's law of heat conduction, or
- Ohm's law of electric charge.

Observation:
Three different physical problems lead to the same mathematical equation!

Heat Conduction or Diffusion:

- Let the function $\rho(\mathbf{x}, t)$ denote a:
  - chemical concentration
  - temperature
  - electro-static potential
- Let the flux $\mathbf{q}$ be proportional to the gradient of ρ (with reversed sign):

$$\mathbf{q}(\mathbf{x}, t) = -a\nabla\rho(\mathbf{x}, t), \quad 0 < a \in \mathbb{R}.$$

- Then:

$$\begin{aligned} \frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla \cdot (-a\nabla\rho(\mathbf{x}, t)) &= 0 \\ \Rightarrow \quad \frac{\partial}{\partial t}\rho(\mathbf{x}, t) &= a\Delta\rho(\mathbf{x}, t). \end{aligned}$$

- With $a := 1$ we obtain the heat equation (or diffusion equation):

$$\frac{\partial}{\partial t}\rho(\mathbf{x}, t) = \Delta\rho(\mathbf{x}, t).$$

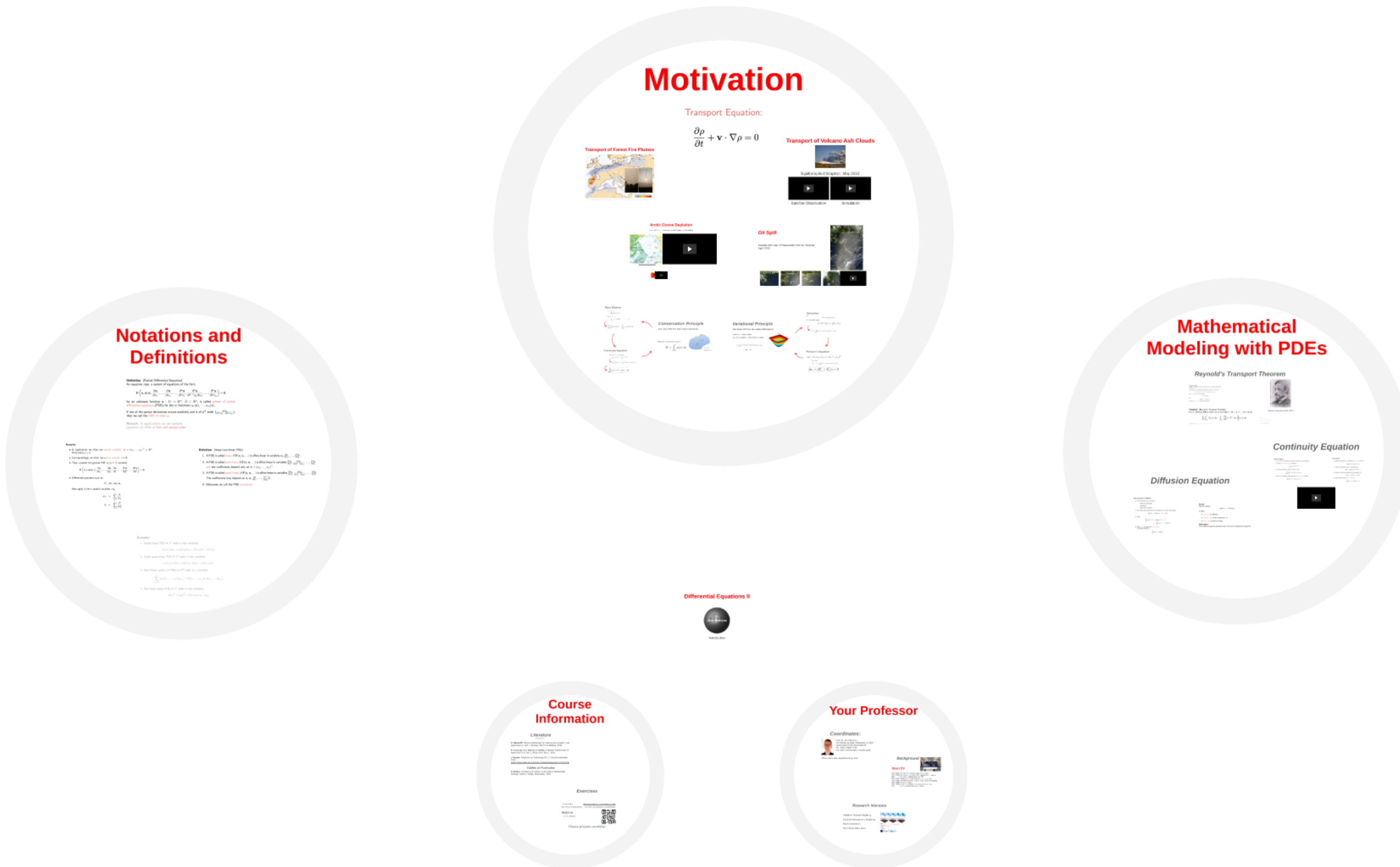
Remark:
The flux relation

$$\mathbf{q}(\mathbf{x}, t) = -a \nabla \rho(\mathbf{x}, t)$$

is called

- Fick's law of diffusion,
- Fourier's law of heat conduction, or
- Ohm's law of electric charge.

Observation:
Three different physical problems lead to the same mathematical equation!


Motivation
Transport Equation:
$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$
Transport of Forest Fire Plumes
Transport of Volcano Ash Clouds
Arctic Ozone Depletion
Oil Spill
Notations and Definitions
Mathematical Modeling with PDEs
Reynold's Transport Theorem
Continuity Equation
Diffusion Equation
Differential Equations II
Course Information
Your Professor