

Differential Equations II



Method of Characteristics

Recapitulation

Definition: (Partial Differential Equation)
An equation resp. a system of equations of the form

$$F\left(x, u(x), \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \dots, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^p u}{\partial x_1 \dots \partial x_n}\right) = 0$$

for an unknown function $u : D \rightarrow \mathbb{R}^m$, $D \subset \mathbb{R}^n$, is called **system of partial differential equations (PDE)** for the m functions $u_1(x), \dots, u_m(x)$.

If one of the partial derivatives occurs explicitly and is of p^{th} order ($\frac{\partial^p u}{\partial x_1 \dots \partial x_n}$), then we call the PDE of **order p** .

Theorem: (Reynold's Transport Theorem)

For an arbitrary differentiable, scalar function $f : D_t \times [0, T] \rightarrow \mathbb{R}$ it holds:

$$\frac{d}{dt} \int_{D_t} f(x, t) \, dx = \int_{D_t} \left[\frac{\partial}{\partial t} f + \nabla \cdot (f v) \right] (x, t) \, dx.$$

Remember: $\operatorname{div} y = \nabla \cdot y$.

Definition: (linear/non-linear PDE)

1. A PDE is called **linear**, if $F(x, u, \dots)$ is affine linear in variables $u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^p u}{\partial x_1 \dots \partial x_n}$.
2. A PDE is called **semi-linear**, if $F(x, u, \dots)$ is affine linear in variables $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial^p u}{\partial x_1 \dots \partial x_n}$ and the coefficients depend only on $x = (x_1, \dots, x_n)^T$.
3. A PDE is called **quasi-linear**, if $F(x, u, \dots)$ is affine linear in variables $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial^p u}{\partial x_1 \dots \partial x_n}$. The coefficients may depend on $x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^{p-1} u}{\partial x_1 \dots \partial x_n}$.
4. Otherwise we call the PDE **non-linear**.

Definition: (Partial Differential Equation)

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$$\mathbf{F} \left(\mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial \mathbf{u}}{\partial x_n}, \dots, \frac{\partial^p \mathbf{u}}{\partial^{p_1} x_1}, \frac{\partial^p \mathbf{u}}{\partial^{p-1} x_1 \partial x_2}, \dots, \frac{\partial^p \mathbf{u}}{\partial^p x_n} \right) = \mathbf{0}$$

for an unknown function $\mathbf{u} : D \rightarrow \mathbb{R}^m$, $D \subset \mathbb{R}^n$, is called **system of partial differential equations** (PDE) for the m functions $u_1(\mathbf{x}), \dots, u_m(\mathbf{x})$.

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Definition: (linear/non-linear PDE)

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The coefficients may depend on $\mathbf{x}, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial^{p-1} \mathbf{u}}{\partial x_n^{p-1}}$.
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Remember: $\operatorname{div} \mathbf{y} = \nabla \cdot \mathbf{y}$.

Preliminaries

Method of Characteristics

Aim: Solution of scalar quasi-linear PDE of 1st order

$$\sum_{i=1}^n a_i(x, u) u_{x_i} = b(x, u), \quad x \in \mathbb{R}^n.$$

Solution method: Method of Characteristics.

Definition: The solution $u(x)$ is called **first integral** of the characteristic system of differential equations.

Remarks: The method of characteristics is a reduction of the PDE to ODEs.

1

Consider first: Homogeneous linear PDE of 1st order

$$\sum_{i=1}^n a_i(x) u_{x_i} = 0, \quad x \in \mathbb{R}^n.$$

Definition: The autonomous system of ordinary differential equations

$$\dot{x}(t) = a(x(t))$$

with $a = (a_1, \dots, a_n)^T$, is called **characteristic system of differential equations** of the homogeneous linear PDE of 1st order.

Observation: With the characteristic ODE we have:

$$\frac{d}{dt} u(x(t)) = \sum_{i=1}^n \frac{d}{dt} x_i(t) \cdot u_{x_i}(x(t)) = \sum_{i=1}^n a_i(x(t)) u_{x_i}(x(t)).$$

Since the right hand side is zero, we obtain:

Remark: The function $u(x)$ is solution of the PDE, iff u is constant along each solution $x(t)$ of the characteristic ODE system, i.e.

$$u(x(t)) = \text{const.}$$

Aim: Solution of scalar quasi-linear PDE of 1st order

$$\sum_{i=1}^n a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u), \quad \mathbf{x} \in \mathbb{R}^n.$$

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Remark: The function $u(\mathbf{x})$ is solution of the PDE, iff u is constant along each solution $\mathbf{x}(t)$ of the characteristic ODE system, i.e.

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Definition: The solution $u(\mathbf{x})$ is called **first integral** of the characteristic system of differentialequations.

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Quasilinear Inhomogeneous PDE

Aim: Application of method of characteristics to inhomogeneous quasi-linear PDE

$$\sum_{i=1}^n a_i(x, u) u_{x_i} = A(x, u), \quad x \in \mathbb{R}^n.$$

Idea: Consider the extended problem

$$\sum_{i=1}^n a_i(x, u) u_{x_i} + A(x, u) u_x = 0, \quad x \in \mathbb{R}^n$$

with the unknown function $U = U(x, u)$ of $(n+1)$ independent variables x and u .

Observation:
In contrast to linear PDEs for quasi-linear PDEs one obtains
no explicit form of solution
and the solution exists possibly only local.

Example: We seek the solution of

$$(1+x)u_x - (1+y)u_y + (z-u)u_z = z.$$

• Extended problem:

$$(1+x)u_x - (1+y)u_y + (z-u)u_z = z.$$

• Characteristic system of ODEs:

$$\begin{aligned} \dot{x} &= 1+x \\ \dot{y} &= -(1+y) \\ \dot{z} &= z-u \\ \dot{u} &= u \end{aligned}$$

• General solution for ODE:

$$\begin{aligned} u(t) &= u_0 e^t \\ y(t) &= u_0^{-1} - 1 \\ z(t) &= (z_0 - u_0) e^t + u_0 \end{aligned}$$

• Eliminate t :

$$z = \frac{z_0 - u_0}{u_0} u + u = (1 + \frac{z_0 - u_0}{u_0}) u = (1 + \frac{z_0}{u_0} - 1) u = \frac{z_0}{u_0} u$$

• Both conditions u and y generate the solution manifold. This yields the **implicit form of the solution**:

$$(1+x)u - (1+y)u = z_0$$

with arbitrary C^1 function $z_0: \mathbb{R}^2 \rightarrow \mathbb{R}$.

Aim: Application of method of characteristics to inhomogeneous quasi-linear PDE

$$\sum_{i=1}^n a_i(\mathbf{x}, u)u_{x_i} = b(\mathbf{x}, u), \quad \mathbf{x} \in \mathbb{R}^n.$$

Idea: Consider the extended problem

$$\sum_{i=1}^n a_i(\mathbf{x}, u)U_{x_i} + b(\mathbf{x}, u)U_u = 0, \quad \mathbf{x} \in \mathbb{R}^n$$

with the unknown function $U = U(\mathbf{x}, u)$ of $(n + 1)$ independent variables \mathbf{x} and u .

Claim: If $U(\mathbf{x}, u)$ is a solution with $U_u \neq 0$, then an implicit form of the solution $u = u(\mathbf{x})$ of the original problem is given by $U(\mathbf{x}, u) = 0$.



A small diagram consisting of a vertical list of mathematical expressions, each preceded by a small arrow pointing to the right. The expressions are: $\sum_{i=1}^n a_i(\mathbf{x}, u)u_{x_i} = b(\mathbf{x}, u)$, $\sum_{i=1}^n a_i(\mathbf{x}, u)U_{x_i} + b(\mathbf{x}, u)U_u = 0$, $U(\mathbf{x}, u) = 0$, and $u = u(\mathbf{x})$.

Proof:

- If $U_u \neq 0$, then one can eliminate $U(\mathbf{x}, u)$ according to the theorem on implicit functions w.r.t. $u(\mathbf{x})$.
- Due to $U(\mathbf{x}, u) = 0$ we have

$$U_{x_i} + U_u u_{x_i} = 0.$$

- Furthermore

$$\sum_{i=1}^n a_i(\mathbf{x}, u) U_{x_i} + b(\mathbf{x}, u) U_u = 0$$

- With the equation above it follows

$$- \left(\sum_{i=1}^n a_i(\mathbf{x}, u) u_{x_i} \right) U_u + b(\mathbf{x}, u) U_u = 0$$

- Finally, for $U_u \neq 0$

$$\sum_{i=1}^n a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u)$$

Example: We seek the solution of

$$(1+x)u_x - (1+y)u_y = y - x.$$

- Extended problem:

$$(1+x)U_x - (1+y)U_y + (y-x)U_u = 0.$$

- Characteristic system of ODEs:

$$\begin{aligned}\dot{x} &= 1+x \\ \dot{y} &= -(1+y) \\ \dot{u} &= y-x\end{aligned}$$

- General solution for ODE:

$$\begin{aligned}x(t) &= c_1 e^t - 1 \\ y(t) &= c_2 e^{-t} - 1 \\ u(t) &= c_3 - c_2 e^{-t} - c_1 e^t\end{aligned}$$

- Eliminate t :

$$e^t = \frac{x+1}{c_1} = \frac{c_2}{y+1} \Rightarrow (x+1)(y+1) = c_1 \cdot c_2 = c \in \mathbb{R}$$

and

$$u = c_3 - (x+1) - (y+1) \Rightarrow u + x + y = d \in \mathbb{R}.$$

- Both constants c and d determine the solution behavior. This yields the **implicit** form of the solution

$$\Phi((x+1)(y+1), (u+x+y)) = 0$$

with arbitrary C^1 -function $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$.

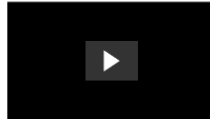
Observation:

In contrast to linear PDEs for quasi-linear PDEs one obtains
no explicit form of solution
and the solution exists possibly only local.

Initial Value Problems

Preliminary Remark. In applications one often considers a time variable t and n spatial variables $\mathbf{x} = (x_1, \dots, x_n)^T$.

Interpretation:
The given initial profile $u_0(\mathbf{x})$ is moved (transported) with constant velocity $\mathbf{a} \in \mathbb{R}^n$ preserving its original shape.



Example:
Use the method of characteristics to solve the transport equation

$$\begin{cases} u_t + \mathbf{a} \cdot \nabla u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \text{ const. } \mathbf{a} \in \mathbb{R}^n \\ u = u_0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Solution: One obtains the form of the solution

$$u(\mathbf{x}, t) = u_0(\mathbf{x} - \mathbf{a}t).$$

2

Preliminary Remark: In applications one often considers a time variable t and n spatial variables $\mathbf{x} = (x_1, \dots, x_n)^\top$.

Definition (Cauchy Problem):

The initial value problem defined on the whole \mathbb{R}^n

$$\begin{cases} u_t + \sum_{i=1}^n a_i(\mathbf{x}, t, u) u_{x_i} = b(\mathbf{x}, t, u) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = u_0 & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

is called **Cauchy Problem**.

Remark (initial conditions):

At time $t = 0$ the **initial condition**

$$u(\mathbf{x}, 0) = u_0(\mathbf{x})$$

is explicitly given.

Example:

Use the method of characteristics to solve the transport equation

$$\begin{cases} u_t + \mathbf{a} \cdot \nabla u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \text{ const. } \equiv \mathbf{a} \in \mathbb{R}^n \\ u = u_0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Solution: One obtains the form of the solution

$$u(\mathbf{x}, t) = u_0(\mathbf{x} - \mathbf{a}t).$$



Check:

With $u(\mathbf{x}, t) = u_0(\mathbf{x} - \mathbf{a}t)$ it holds

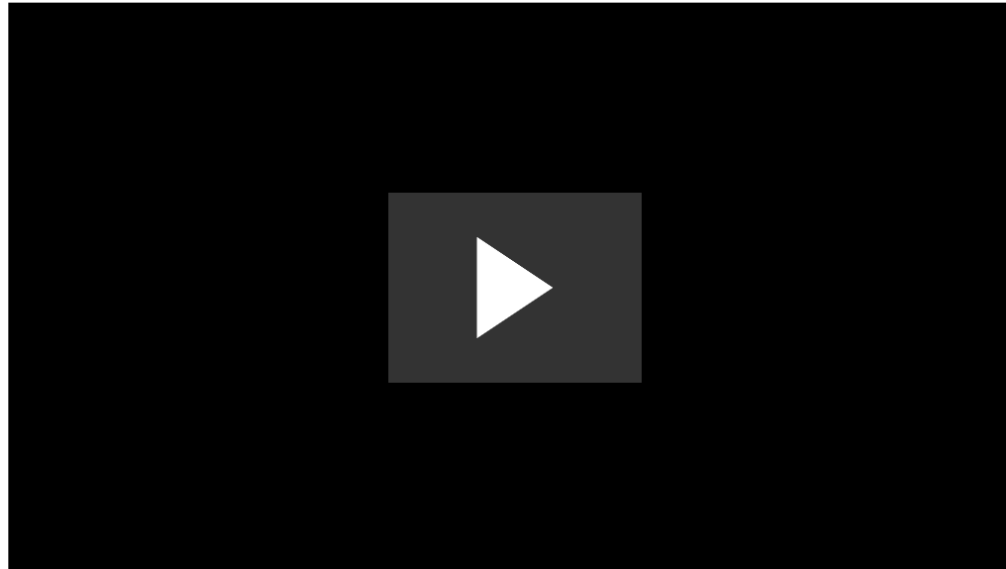
$$u_t(\mathbf{x}, t) = -\mathbf{a} \cdot \nabla u_0, \quad \nabla u(\mathbf{x}, t) = \nabla u_0$$

So

$$u_t + \mathbf{a} \cdot \nabla u = 0.$$

Interpretation:

The given initial profile $u_0(\mathbf{x})$ is moved (transported) with constant velocity $\mathbf{a} \in \mathbb{R}^n$ preserving its original shape.



Preliminaries Method of Characteristics

File: Section 1 of your book: PDE of 2nd order

Mathematical method: Method of Characteristics

Mathematical software: Maple

Mathematical software: Mathematica

Mathematical software: MATLAB

Mathematical software: Python

Mathematical software: R

Mathematical software: Julia

Mathematical software: Octave

Mathematical software: Scilab

Mathematical software: SageMath

Mathematical software: SymPy

Mathematical software: Maxima

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