Differential Equations II

Method of Characteristics

Recapitulation

Definition: (Partial Differential Equation)
An equation resp. a system of equations of the form

$$\mathbf{F}\left(\mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial \mathbf{u}}{\partial x_n}, \dots, \frac{\partial^p \mathbf{u}}{\partial^p x_1}, \frac{\partial^p \mathbf{u}}{\partial^{p-1} x_1 \partial x_2}, \dots, \frac{\partial^p \mathbf{u}}{\partial^p x_n}\right) = \mathbf{0}$$

for an unknown function $\mathbf{u} : D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$, is called system of partial differential equations (PDE) for the m functions $u_1(\mathbf{x}), \dots, u_m(\mathbf{x})$.

If one of the partial derivatives occurs explicitly and is of p^{th} order ($\frac{\partial^p \mathbf{u}}{\partial^{p_1} x_1 \cdots \partial^{p_n} x_n}$), then we call the PDE of order p.

Theorem: (Reynold's Transport Theorem)
For an arbitrary differeniable, scalar function $f : D_t \times [0, T] \to \mathbb{R}$ it holds:

$$\frac{d}{dt} \int_{D_t} f(\mathbf{x}, t)\, d\mathbf{x} = \int_{D_t} \left[\frac{\partial}{\partial t} f + \nabla \cdot (f \mathbf{v})\right] (\mathbf{x}, t)\, d\mathbf{x}.$$

Remember: $\operatorname{div} \mathbf{y} = \nabla \cdot \mathbf{y}$.

Definition: (linear/non-linear PDE)

1. A PDE is called linear, if $\mathbf{F}(\mathbf{x}, \mathbf{u}, \dots)$ is affine linear in variables $\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$.
2. A PDE is called semi-linear, if $\mathbf{F}(\mathbf{x}, \mathbf{u}, \dots)$ is affine linear in variables $\frac{\partial^p \mathbf{u}}{\partial x_1^p}, \frac{\partial^p \mathbf{u}}{\partial x_1^{p-1} \partial x_2}, \dots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$ and the coefficients depend only on $\mathbf{x} = (x_1, \dots, x_n)^\top$.
3. A PDE is called quasi-linear, if $\mathbf{F}(\mathbf{x}, \mathbf{u}, \dots)$ is affine linear in variables $\frac{\partial^p \mathbf{u}}{\partial x_1^p}, \frac{\partial^p \mathbf{u}}{\partial x_1^{p-1} \partial x_2}, \dots, \frac{\partial^p \mathbf{u}}{\partial x_n^p}$. The coefficients may depend on $\mathbf{x}, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial^{p-1} \mathbf{u}}{\partial x_n^{p-1}}$.
4. Otherwise we call the PDE non-linear.

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Preliminaries
Method of Characteristics

Aim: Solution of scalar quasi-linear PDE of 1st order

$$\sum_{i=1}^{n} a_i(\mathbf{x},u)u_{x_i} = b(\mathbf{x},u), \quad \mathbf{x} \in \mathbb{R}^n.$$

Solution method: Method of Characteristics.

Definition: The solution $u(\mathbf{x})$ is called first integral of the characteristic system of differentialequations.

Remarks: The method of characteristics is a reduction of the PDE to ODEs.

Consider first: Homogeneous linear PDE of 1st order

$$\sum_{i=1}^{n} a_i(\mathbf{x})u_{x_i} = 0, \quad \mathbf{x} \in \mathbb{R}^n.$$

Definition: The autonomous system of ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t))$$

with $\mathbf{a} = (a_1, \ldots, a_n)^\top$, is called characteristic system of differential equations of the homogeneous lineare PDE of 1st order.

Observation: With the characteristic ODE we have:

$$\frac{d}{dt}u(\mathbf{x}(t)) = \sum_{i=1}^{n} \frac{d}{dt}x_i(t) \cdot u_{x_i}(\mathbf{x}(t)) = \sum_{i=1}^{n} a_i(\mathbf{x}(t))u_{x_i}(\mathbf{x}(t)).$$

Since the right hand side is zero, we obtain:

Remark: The function $u(\mathbf{x})$ is solution of the PDE, iff u is constant along each solution $\mathbf{x}(t)$ of the characteristic ODE system, i.e.

$$u(\mathbf{x}(t)) \equiv \text{const.}$$

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Quasilinear Inhomogeneous PDE

Aim: Application of method of characteristics to inhomogeneous quasi-linear PDE

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u), \quad \mathbf{x} \in \mathbb{R}^n.$$

Idea: Consider the extended problem

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) U_{x_i} + b(\mathbf{x}, u) U_u = 0, \quad \mathbf{x} \in \mathbb{R}^n$$

with the unkown function $U = U(\mathbf{x}, u)$ of $(n+1)$ independent variables $\mathbf{x}$ and u.

Observation:
In contrast to linear PDEs for quasi-linear PDEs one obtains
no explicit form of solution
and the solution exists possibly only local.

Example: We seek the solution of

$$(1+x)u_x - (1+y)u_y = y - x.$$

- Extended problem:

$$(1+x)U_x - (1+y)U_y + (y-x)U_u = 0.$$

- Characteristic system of ODEs:

$$\begin{aligned} \dot{x} &= 1 + x \\ \dot{y} &= -(1+y) \\ \dot{u} &= y - x \end{aligned}$$

- General solution for ODE:

$$\begin{aligned} x(t) &= c_1 e^t - 1 \\ y(t) &= c_2 e^{-t} - 1 \\ u(t) &= c_3 - c_2 e^{-t} - c_1 e^t \end{aligned}$$

- Eliminate t:

$$e^t = \frac{x+1}{c_1} = \frac{c_2}{y+1} \quad \Rightarrow \quad (x+1)(y+1) = c_1 \cdot c_2 = c \in \mathbb{R}$$

and

$$u = c_3 - (x+1) - (y+1) \quad \Rightarrow \quad u + x + y = d \in \mathbb{R}.$$

- Both constants c and d determine the solution behavior. This yields the implicit form of the solution

$$\Phi((x+1)(y+1), (u+x+y)) = 0$$

with arbitrary C^1-function $\Phi : \mathbb{R}^2 \to \mathbb{R}$.

Aim: Application of method of characteristics to inhomogeneous quasi-linear PDE

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u), \quad \mathbf{x} \in \mathbb{R}^n.$$

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Claim: If $U(\mathbf{x}, u)$ is a solution with $U_u \neq 0$, then an implicit form of the solution $u = u(\mathbf{x})$ of the original problem is given by $U(\mathbf{x}, u) = 0$.

Proof:

- If $U_u \neq 0$, then one can eliminate $U(\mathbf{x}, u)$ according to the theorem on implicit functions w.r.t. $u(\mathbf{x})$.
- Due to $U(\mathbf{x}, u) = 0$ we have

$$U_{x_i} + U_u u_{x_i} = 0.$$

- Furthermore

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) U_{x_i} + b(\mathbf{x}, u) U_u = 0$$

- With the equation above it follows

$$-\left(\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i}\right) U_u + b(\mathbf{x}, u) U_u = 0$$

- Finally, for $U_u \neq 0$

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u)$$

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Initial Value Problems

Preliminary Remark: In applications one often considers a time variable t and n spatial variables $\mathbf{x} = (x_1, \ldots, x_n)^\top$.

Interpretation:
The given initial profile $u_0(\mathbf{x})$ is moved (transported) with constant velocity $\mathbf{a} \in \mathbb{R}^n$ preserving its original shape.

Example:
Use the method of characteristics to solve the transport equation

$$\begin{cases} u_t + \mathbf{a} \cdot \nabla u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \ \text{const.} \equiv \mathbf{a} \in \mathbb{R}^n \\ u = u_0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Solution: One obtains the form of the solution

$$u(\mathbf{x}, t) = u_0(\mathbf{x} - \mathbf{a}t).$$

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Definition (Cauchy Problem):
The initial value problem defined on the whole $\mathbb{R}^n$

$$\begin{cases} u_t + \sum_{i=1}^n a_i(\mathbf{x}, t, u) u_{x_i} = b(\mathbf{x}, t, u) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = u_0 & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

is called Cauchy Problem.

Remark (initial conditions):
At time $t = 0$ the initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x})$$

is explicitly given.

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Use the method of characteristics to solve the transport equation

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Solution: One obtains the form of the solution

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$$u(\mathbf{x}, t) = u_0(\mathbf{x} - \mathbf{a}t).$$

Check:
With $u(\mathbf{x}, t) = u_0(\mathbf{x} - \mathbf{a}t)$ it holds

$$u_t(\mathbf{x}, t) = -\mathbf{a}\nabla u_0, \quad \nabla u(\mathbf{x}, t) = \nabla u_0$$

So

$$u_t + \mathbf{a} \cdot \nabla u = 0.$$

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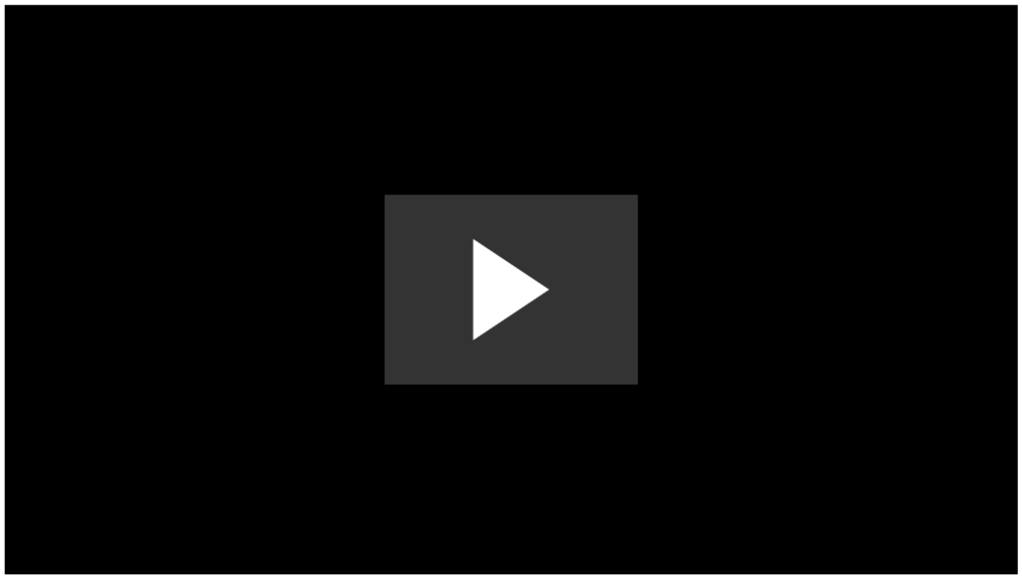

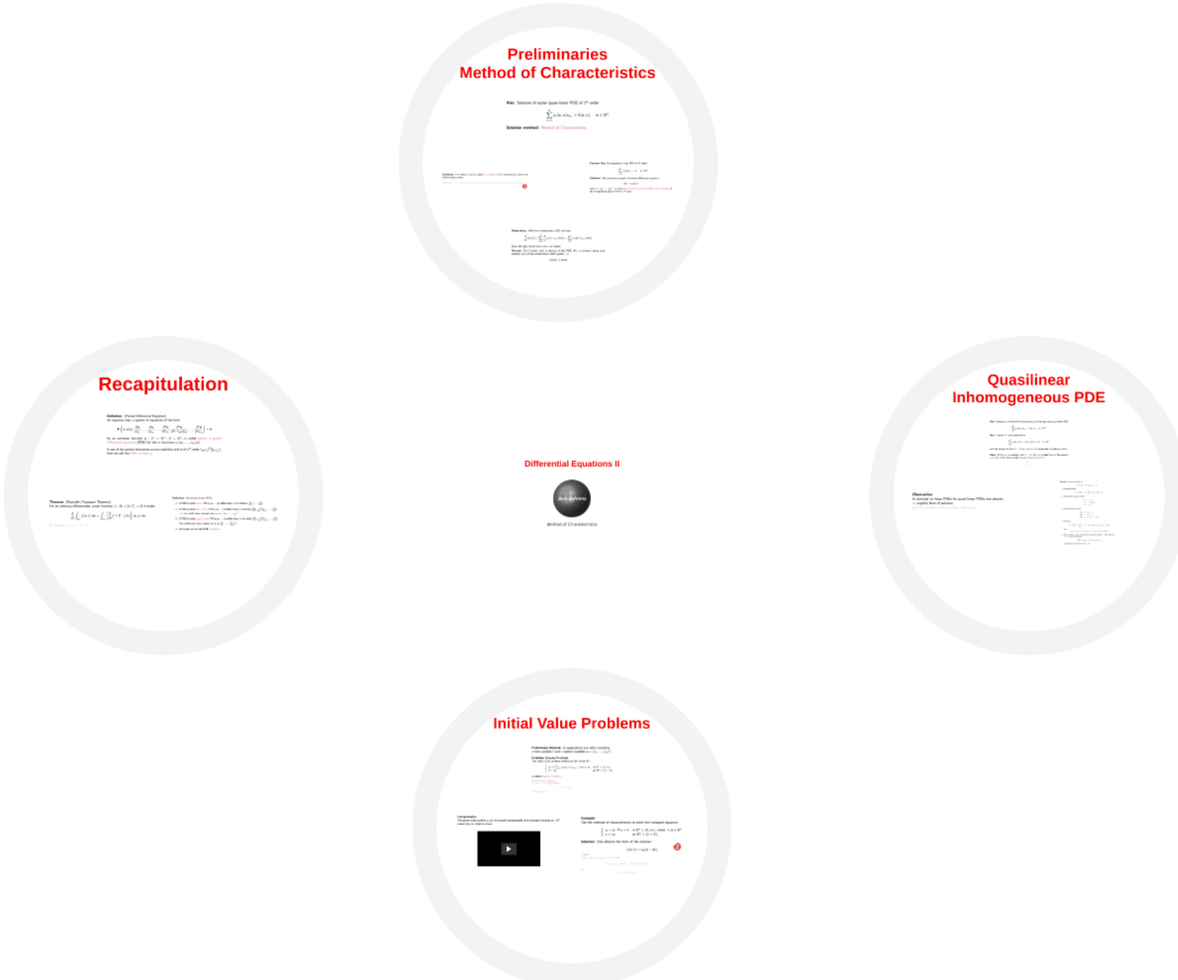
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