Differential Equations II

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Method of Characteristics for
non-linear Equations

Recap

Definition: (Characteristic System of Differential Equations)
Let the scalar linear homogeneous PDE of 1st order

$$\sum_{i=1}^{n} a_i(\mathbf{x}) u_{x_i} = 0, \quad \mathbf{x} \in \mathbb{R}^n$$

be given. Then the autonomous system of ODEs

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t))$$

is the characteristic system of differential equations (CSDE) of the PDE.
Solution methods, using the CSDE are called Methods of Characteristics.

Remarks: (Method of Characteristics)

- By means of the characteristic system of differential equations (CSDE) we obtain:
$$\frac{d}{dt} u(\mathbf{x}(t)) = \sum_{i=1}^{n} a_i(\mathbf{x}(t)) u_{x_i}(\mathbf{x}(t)) = 0$$
and thus $u(\mathbf{x}(t)) = \text{const.}$. This solution is called first integral.
- This solution methods can be applied to quasi-linear inhomogeneous PDEs
$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u), \quad \mathbf{x} \in \mathbb{R}^n$$
by considering the extended problem
$$\sum_{i=1}^{n} a_i(\mathbf{x}, u) U_{x_i} + b(\mathbf{x}, u) U_u = 0$$
for $U = U(\mathbf{x}, u)$.

Definition: (Cauchy Problem)
For time-dependent equations with time variable $t \in I = [0, \infty[$ and spatial variables $\mathbf{x} \in \mathbb{R}^n$ consider the initial value problem defined on the whole of $\mathbb{R}^n \times I$

$$\begin{aligned} u_t + \sum_{i=1}^{n} a_i(\mathbf{x}, t, u) u_{x_i} &= b(\mathbf{x}, t, u) \quad \text{in } \mathbb{R}^n \times I \\ u &= u_0 \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{aligned}$$

This problem is called Cauchy-Problem.

Example: (Transport Equation)
The transport equation

$$\begin{aligned} u_t + \mathbf{a} \cdot \nabla u &= u_t + \sum_{i=1}^{n} a_i u_{x_i} = 0 \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}) \end{aligned}$$

with $(\mathbf{x}, t) \in \mathbb{R}^n \times I$ has the CSDE

$$\dot{t}(\tau) = 1, \; \dot{x}_1(\tau) = a_1, \ldots, \dot{x}_n(\tau) = a_n.$$

With $t = \tau$ the n equations $\dot{x}_i(t) = a_i$ remain. Thus, the solution is a linear system of the form

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{a} \cdot t.$$

Elimination to $\mathbf{x}_0$ and substitution of initial values yields the solution

$$u(\mathbf{x}, t) = u_0(\mathbf{x} - \mathbf{a}t).$$

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Burgers Equation

Example:
In comparison to the transport equation we increase the complexity slightly, by choosing $a = tx$. Consider this equation now in $\mathbb{R} \times I$:

$$\begin{aligned} u_t + txu_x &= 0 && \text{in } \mathbb{R}\times]0,\infty[\\ u(x,0) &= \sin(x) && \text{auf } \mathbb{R} \times \{t=0\} \end{aligned}$$

- Characteristic equation: $\dot{x} = tx$, $x(0) = x_0$.
- Solution of characteristic equation: $x(t) = x_0 \exp\left(\frac{t^2}{2}\right)$
- Solution of IVP: $u(x,t) = \sin\left[x \exp\left(-\frac{t^2}{2}\right)\right]$.

Example: (non-linear scalar conservation law)
The folowing Cauchy problem represents a non-linear scalar conservation law in one spatial dimension.

$$\begin{aligned} u_t + f(u)_x &= 0 && \text{in } \mathbb{R}\times]0,\infty[\\ u &= u_0 && \text{auf } \mathbb{R} \times \{t=0\} \end{aligned}$$

- $f = f(u)$ given is called flux function.
- This PDE is quasi-linear, since it can be written as

$$u_t + a(u)u_x = 0$$

  with $a(u) = f'(u)$.
- $a(u)$ can be called local dispersion velocity. 1

Example: (Burgers' Equation)
Burgers' Gleichung is given by the flux function $f(u) = \frac{u^2}{2}$, resp. by the Cauchy problem


Jan M. Burgers (1895-1981)

$$\begin{aligned} u_t + uu_x &= 0 && \text{in } \mathbb{R}\times]0,\infty[\\ u &= u_0 && \text{on } \mathbb{R} \times \{t=0\} \end{aligned}$$

- The solution is given by $u(t) = x_0 + tu_0(x_0)$.
- If u_0 is given by

$$u_0(x) = \begin{cases} 1 & : \quad x \le 0 \\ 1-x & : \quad 0 < x < 1 \\ 0 & : \quad x \ge 1 \end{cases}$$

  then $x(t)$ develops a singularity for $t \to 1$.
- A classical solution of Burgers' equation exists only locally for $0 \le t < 1$.
- The local solution for $t \in [0,1[$ is:

$$u(x,t) = \begin{cases} 1 & : \quad x < 1 \\ \frac{(1-x)}{(1-t)} & : \quad 0 \le t \le x \le 1 \\ 0 & : \quad x > 1 \end{cases}$$

Conclusion: The scalar conservation law given by the Cauchy problem

$$\begin{aligned} u_t + f(u)_x &= 0 && \text{in } \mathbb{R}\times]0,\infty[\\ u &= u_0 && \text{auf } \mathbb{R} \times \{t=0\} \end{aligned}$$

in general does not have a global solution.

Question: what happens for $t \ge 1$, i.e. behind the singularity?

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http://www.inghist.nl/Onderzoek/Projecten/BWN/lemmata/bwn5/burgers

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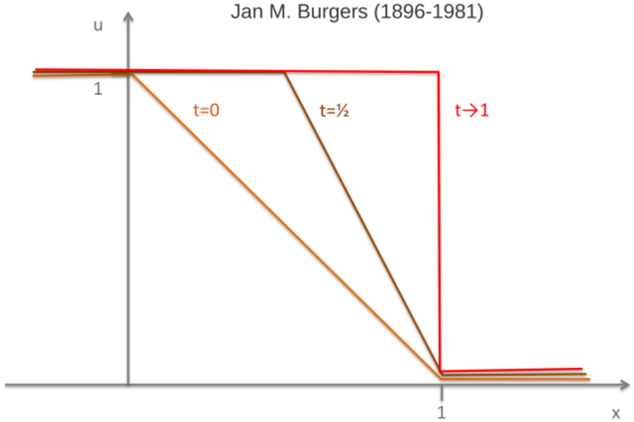

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in general does not have a global solution.

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General Scalar Conservation Law

Definition: (compact support)
The support of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set

$$\mathrm{supp}(f) := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

If $\mathrm{supp}(f)$ is a compact set, then we have a function with compact support.

2

Proposition: (Rankine-Hugoniot condition)
If $x = s(t)$ is a shock front of a shock wave solution of $u_t + f(u)_x = 0$, then for the shock speed $\dot{s}(t)$ the Rankine-Hugoniot condition holds:

$$\dot{s} = \frac{[f]}{[u]} = \frac{f(u(s(t)^-, t)) - f(u(s(t)^+, t))}{u(s(t)^-, t) - u(s(t)^+, t)} = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

4

Remark: (test function)

- Let $v : \mathbb{R} \times [0\infty[\to \mathbb{R}$ differentiable function with compact support.
- Consider a scalar conservation law $u_t + f(u)_x = 0$, multiply with v and integrate (by parts):

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (u_t + f(u)_x) v \; dx dt \\ &= -\int_0^\infty \int_{-\infty}^\infty u v_t \; dx dt - \int_{-\infty}^\infty u(x,0) v(x,0) \; dx - \int_0^\infty \int_{-\infty}^\infty f(u) v_x \; dx dt. \end{aligned}$$

- Initial conditions $u(x,0) = u_0(x)$ yield

$$\int_0^\infty \int_{-\infty}^\infty (u v_t + f(u) v_x) \; dx dt + \int_{-\infty}^\infty u_0(x) v(x,0) \; dx = 0.$$

- Such function v is called test function.

Definition: (shock wave solution)
A shock wave solution u is a weak solution of the conservation law

$$u_t + f(u)_x = 0$$

if a shock front $x = s(t)$, $s \in C^1$ exists, such that u is a classical solution for each $x < s(t)$ and $x > s(t)$ and u has a jump at $x = s(t)$ with height

$$[u](t) = u(s(t)^+, t) - u(s(t)^-, t) = u_r - u_l.$$

$\dot{s}(t)$ is called shock speed.

Definition: (weak solution)
A function $u \in L^\infty(\mathbb{R} \times [0, \infty[)$ is called integral solution or weak solution of the conservation law $u_t + f(u)_x = 0$, if for all test functions v:

$$\int_0^\infty \int_{-\infty}^\infty (u v_t + f(u) v_x) \; dx dt + \int_{-\infty}^\infty u_0(x) v(x,0) \; dx = 0.$$

Definition: (Riemann problem)
The initial value problem

$$\begin{aligned} u_t + f(u)_x &= 0 \quad \text{in } \mathbb{R} \times]0, \infty[\\ u &= u_0 \quad \text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

with discontinuous initial conditions

$$u_0(x) = \begin{cases} u_l & : x \leq 0 \\ u_r & : x > 0 \end{cases}$$

is called Riemann problem for the scalar conservation law. 3

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Remarks:

- Many differentiable functions have compact support.
- They are important in theory and numerics of PDEs.

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Remarks:

- A weak solution needs not be differentiable function!
- It even can have a jump.

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https://upload.wikimedia.org/wikipedia/commons/5/55/William_Rankine_1870s3.jpg

W.J.Macquorn Rankine (1820-1872)

Pierre-Henri Hugoniot (1851-1887)

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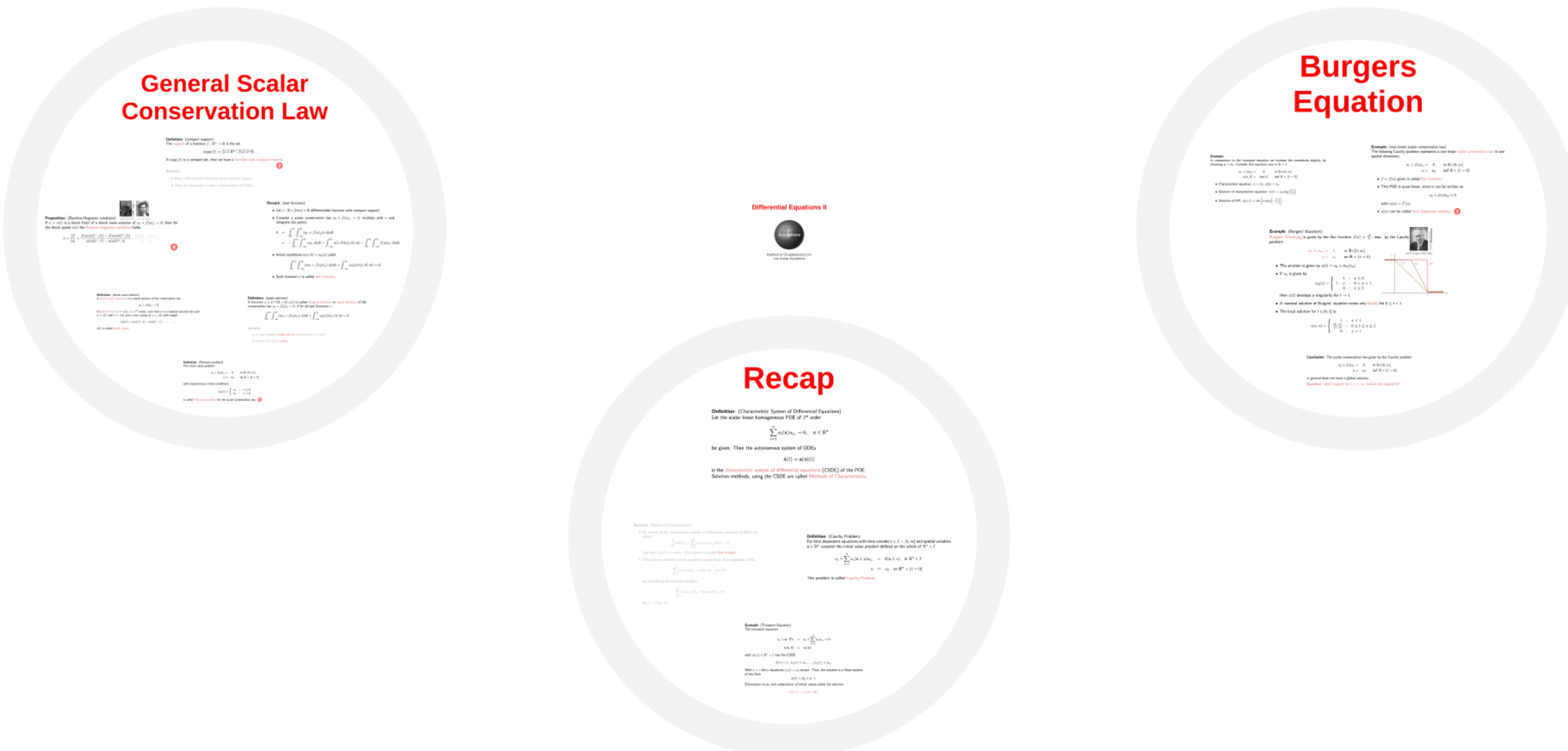
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