

Differential Equations II



Method of Characteristics for
non-linear Equations

Recap

Definition: (Characteristic System of Differential Equations)
Let the scalar linear homogeneous PDE of 1st order

$$\sum_{i=1}^n a_i(\mathbf{x}) u_{x_i} = 0, \quad \mathbf{x} \in \mathbb{R}^n$$

be given. Then the autonomous system of ODEs

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t))$$

is the **characteristic system of differential equations (CSDE)** of the PDE. Solution methods, using the CSDE are called **Methods of Characteristics**.

Remarks: (Method of Characteristics)

- By means of the characteristic system of differential equations (CSDE) we obtain

$$\frac{d}{dt} u(\mathbf{x}(t)) = \sum_{i=1}^n a_i(\mathbf{x}(t)) u_{x_i}(\mathbf{x}(t)) = 0$$

and thus $u(\mathbf{x}(t)) = \text{const.}$. This solution is called **first integral**.

- This solution methods can be applied to quasi-linear inhomogeneous PDEs

$$\sum_{i=1}^n a_i(\mathbf{x}, u) u_{x_i} = b(\mathbf{x}, u), \quad \mathbf{x} \in \mathbb{R}^n$$

by considering the extended problem

$$\sum_{i=1}^n a_i(\mathbf{x}, u) \tilde{u}_{x_i} + b(\mathbf{x}, u) \tilde{u}_u = 0$$

for $\tilde{t} = t(\mathbf{x}, u)$.

Definition: (Cauchy Problem)

For time-dependent equations with time variable $t \in I = [0, \infty[$ and spatial variables $\mathbf{x} \in \mathbb{R}^n$ consider the initial value problem defined on the whole of $\mathbb{R}^n \times I$

$$u_t + \sum_{i=1}^n a_i(\mathbf{x}, t, u) u_{x_i} = b(\mathbf{x}, t, u) \quad \text{in } \mathbb{R}^n \times I$$

$$u = u_0 \quad \text{on } \mathbb{R}^n \times \{t = 0\}$$

This problem is called **Cauchy Problem**.

Example: (Transport Equation)

The transport equation

$$u_t + \mathbf{a} \cdot \nabla u = u_t + \sum_{i=1}^n a_i u_{x_i} = 0$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x})$$

with $(\mathbf{x}, t) \in \mathbb{R}^n \times I$ has the CSDE

$$\dot{t}(\tau) = 1, \quad \dot{x}_i(\tau) = a_i, \dots, \dot{x}_n(\tau) = a_n$$

With $t = \tau$ the n equations $\dot{x}_i(\tau) = a_i$ remain. Thus, the solution is a linear system of the form

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{a} \cdot t.$$

Elimination to x_0 and substitution of initial values yields the solution

$$u(\mathbf{x}, t) = u_0(\mathbf{x} - \mathbf{a}t).$$

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$$\sum_{i=1}^n a_i(\mathbf{x}, u)U_{x_i} + b(\mathbf{x}, u)U_u = 0$$

for $U = U(\mathbf{x}, u)$.

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Burgers Equation

Example:

In comparison to the transport equation we increase the complexity slightly, by choosing $a = fx$. Consider this equation now in $\mathbb{R} \times I$.

$$\begin{aligned} u_t + t u_x &= 0 && \text{in } \mathbb{R} \times]0, \infty[\\ u(x, 0) &= \sin(x) && \text{auf } \mathbb{R} \times \{t=0\} \end{aligned}$$

- Characteristic equation: $\dot{x} = tx, x(0) = x_0$.
- Solution of characteristic equation: $x(t) = x_0 \exp\left(\frac{t^2}{2}\right)$
- Solution of IVP: $u(x, t) = \sin\left[\exp\left(-\frac{t^2}{2}\right)\right]$.

Example: (non-linear scalar conservation law)

The following Cauchy problem represents a non-linear scalar conservation law in one spatial dimension.

$$\begin{aligned} u_t + f(u)_x &= 0 && \text{in } \mathbb{R} \times]0, \infty[\\ u &= u_0 && \text{auf } \mathbb{R} \times \{t=0\} \end{aligned}$$

- $f = f(u)$ given is called **flux function**.
- This PDE is quasi-linear, since it can be written as

$$u_t + a(u)u_x = 0$$

with $a(u) = f'(u)$.

- $a(u)$ can be called **local dispersion velocity**. 1

Example: (Burgers' Equation)

Burgers' Gleichung is given by the flux function $f(u) = \frac{u^2}{2}$, resp. by the Cauchy problem

$$\begin{aligned} u_t + uu_x &= 0 && \text{in } \mathbb{R} \times]0, \infty[\\ u &= u_0 && \text{on } \mathbb{R} \times \{t=0\} \end{aligned}$$

- The solution is given by $u(t) = x_0 + t u_0(x_0)$.

- If u_0 is given by

$$u_0(x) = \begin{cases} 1 & : x \leq 0 \\ 1-x & : 0 < x < 1 \\ 0 & : x \geq 1 \end{cases}$$

then $x(t)$ develops a singularity for $t \rightarrow 1$.

- A classical solution of Burgers' equation exists only **locally** for $0 \leq t < 1$.

- The local solution for $t \in]0, 1[$ is:

$$u(x, t) = \begin{cases} 1 & : x < 1 \\ \frac{1-x}{1-t} & : 0 \leq t \leq x \leq 1 \\ 0 & : x > 1 \end{cases}$$

Conclusion: The scalar conservation law given by the Cauchy problem

$$\begin{aligned} u_t + f(u)_x &= 0 && \text{in } \mathbb{R} \times]0, \infty[\\ u &= u_0 && \text{auf } \mathbb{R} \times \{t=0\} \end{aligned}$$

in general does not have a global solution.

Question: what happens for $t \geq 1$, i.e. behind the singularity?



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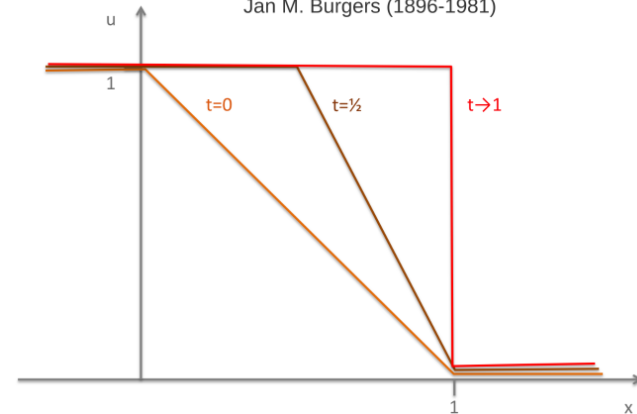
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Jan M. Burgers (1896-1981)



<http://www.mathis.tu-berlin.de/~burgers/>
 Projekt: Burgers' Gleichung

Conclusion: The scalar conservation law given by the Cauchy problem

$$\begin{aligned} u_t + f(u)_x &= 0 && \text{in } \mathbb{R} \times]0, \infty[\\ u &= u_0 && \text{auf } \mathbb{R} \times \{t = 0\} \end{aligned}$$

in general does not have a global solution.

Question: what happens for $t \geq 1$, i.e. behind the singularity?

General Scalar Conservation Law

Definition (compact support)
 The support of a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is the set
 $\text{supp}(f) := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$.

If $\text{supp}(f)$ is a compact set, then we have a function with compact support. **2**



Proposition (Rankine-Hugoniot condition)
 If $x = s(t)$ is a shock front of a shock wave solution of $u_t + f(u)_x = 0$, then for the shock speed $s(t)$ the Rankine-Hugoniot condition holds:

$$s = \frac{[f]}{[u]} = \frac{f(u(s(t)^-, t)) - f(u(s(t)^+, t))}{u(s(t)^-, t) - u(s(t)^+, t)} = \frac{f(u_1) - f(u_2)}{u_1 - u_2} \quad \mathbf{4}$$

Definition (shock wave solution)
 A shock wave solution u is a weak solution of the conservation law
 $u_t + f(u)_x = 0$

if a shock front $x = s(t)$, $s \in C^1$ exists, such that u is a classical solution for each $x < s(t)$ and $x > s(t)$ and u has a jump at $x = s(t)$ with height

$$[u] = u(s(t)^+, t) - u(s(t)^-, t) = u_2 - u_1.$$

$s(t)$ is called shock speed.

Remark (test function)

- Let $\varphi: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ differentiable function with compact support.
- Consider a scalar conservation law $u_t + f(u)_x = 0$, multiply with φ and integrate (by parts):

$$0 = \int_{-\infty}^{\infty} \int_0^{\infty} (u_t + f(u)_x) \varphi \, dx dt = - \int_0^{\infty} \int_{-\infty}^{\infty} u \varphi_x \, dx dt - \int_{-\infty}^{\infty} u(x, 0) \varphi(x, 0) \, dx + \int_0^{\infty} \int_{-\infty}^{\infty} f(u) \varphi_x \, dx dt.$$

- Initial conditions $u(x, 0) = u_0(x)$ yield

$$\int_{-\infty}^{\infty} \int_0^{\infty} (u_t + f(u) \varphi_x) \, dx dt + \int_{-\infty}^{\infty} u_0(x) \varphi(x, 0) \, dx = 0.$$

- Such function φ is called test function.

Definition (weak solution)
 A function $u \in L^1_{loc}(\mathbb{R} \times [0, \infty))$ is called integral solution or weak solution of the conservation law $u_t + f(u)_x = 0$, if for all test functions φ :

$$\int_0^{\infty} \int_{-\infty}^{\infty} (u_t + f(u) \varphi_x) \, dx dt + \int_{-\infty}^{\infty} u_0(x) \varphi(x, 0) \, dx = 0.$$

Definition (Riemann problem)

The initial value problem
 $u_t + f(u)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$
 $u = u_0 \quad \text{on } \mathbb{R} \times \{t = 0\}$

with discontinuous initial conditions

$$u_0(x) = \begin{cases} u_1 & x \leq 0 \\ u_2 & x > 0 \end{cases}$$

is called Riemann problem for the scalar conservation law. **5**

Definition: (compact support)

The **support** of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the set

$$\text{supp}(f) := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

If $\text{supp}(f)$ is a compact set, then we have a **function with compact support**.

2

Remarks:

- Many differentiable functions have compact support.
- They are important in theory and numerics of PDEs.

Remark: (test function)

- Let $v : \mathbb{R} \times [0, \infty[\rightarrow \mathbb{R}$ differentiable function with compact support.
- Consider a scalar conservation law $u_t + f(u)_x = 0$, multiply with v and integrate (by parts):

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (u_t + f(u)_x) v \, dx dt \\ &= - \int_0^\infty \int_{-\infty}^\infty u v_t \, dx dt - \int_{-\infty}^\infty u(x, 0) v(x, 0) \, dx - \int_0^\infty \int_{-\infty}^\infty f(u) v_x \, dx dt. \end{aligned}$$

- Initial conditions $u(x, 0) = u_0(x)$ yield

$$\int_0^\infty \int_{-\infty}^\infty (u v_t + f(u) v_x) \, dx dt + \int_{-\infty}^\infty u_0(x) v(x, 0) \, dx = 0.$$

- Such function v is called **test function**.

Definition: (weak solution)

A function $u \in L^\infty(\mathbb{R} \times [0, \infty[)$ is called **integral solution** or **weak solution** of the conservation law $u_t + f(u)_x = 0$, if for all test functions v :

$$\int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x) dxdt + \int_{-\infty}^\infty u_0(x)v(x, 0) dx = 0.$$

Remarks:

- A weak solution **needs not be** differentiable function!
- It even can have a **jump**.

Definition: (Riemann problem)

The initial value problem

$$\begin{aligned} u_t + f(u)_x &= 0 && \text{in } \mathbb{R} \times]0, \infty[\\ u &= u_0 && \text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

with discontinuous initial conditions

$$u_0(x) = \begin{cases} u_l & : x \leq 0 \\ u_r & : x > 0 \end{cases}$$

is called **Riemann problem** for the scalar conservation law. **3**

Definition: (shock wave solution)

A **shock wave solution** u is a weak solution of the conservation law

$$u_t + f(u)_x = 0$$

if a **shock front** $x = s(t)$, $s \in C^1$ exists, such that u is a classical solution for each $x < s(t)$ and $x > s(t)$ and u has a jump at $x = s(t)$ with height

$$[u](t) = u(s(t)^+, t) - u(s(t)^-, t) = u_r - u_l.$$

$\dot{s}(t)$ is called **shock speed**.



W.J. Macquorn Rankine (1820-1872)



Pierre-Henri Hugoniot (1851-1887)

Proposition: (Rankine-Hugoniot condition)

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4

General Scalar Conservation Law

Definition (Conservation Law)
 The integral of a function $f(x,t)$ over a fixed interval $[a,b]$ is the total amount of f in that interval.

Example
 If $f(x,t)$ is a conserved quantity, then the total amount of f in a fixed interval $[a,b]$ is constant in time.



Proposition (Conservation Law)
 If $f(x,t)$ is a conserved quantity, then the total amount of f in a fixed interval $[a,b]$ is constant in time.

$$\frac{d}{dt} \int_a^b f(x,t) dx = - \int_a^b \partial_x (f v) dx + \int_a^b S dx$$

Definition (Conservation Law)
 A function $f(x,t)$ is a conserved quantity if it satisfies the conservation law.

Example
 The function $f(x,t) = x^2 + t^2$ is a conserved quantity.

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be given. Then the autonomous system of ODEs

$$\dot{x}_i = a_i(x)$$

is the **characteristic system of differential equations (CSDE)** of the PDE. Solution methods, using the CSDE are called **Method of Characteristics**.

Example (Characteristic System)
 For the PDE $\partial_x u + \partial_y u = 0$, the characteristic system is

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 1 \end{cases}$$

The solution is $x = t + x_0, y = t + y_0$.

The general solution is $u(x,y) = f(x-y)$.

Cauchy Problem

For some domain Ω in \mathbb{R}^n with boundary $\partial\Omega$, let $f: \partial\Omega \rightarrow \mathbb{R}$ be a given function. The **Cauchy problem** for the PDE

$$\sum_{i=1}^n a_i(x) \frac{\partial u}{\partial x_i} = 0$$

is to find a function $u: \Omega \rightarrow \mathbb{R}$ satisfying the PDE and the boundary condition $u|_{\partial\Omega} = f$.

The problem is called **Cauchy Problem**.

Example (Cauchy Problem)

For the PDE $\partial_x u + \partial_y u = 0$, let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a given function. The Cauchy problem is to find a function $u: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying the PDE and the boundary condition $u(x,0) = f(x)$.

The solution is $u(x,y) = f(x-y)$.

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