

Recap

Example (Riemann-Hugoniot Condition for Burgers' Equation)
 Consider Burgers' Equation $u_t + u u_x = 0$ in $\mathbb{R} \times (0, \infty)$ and $u(x, 0) = u_0(x)$. Let $u_0(x) = \begin{cases} u_1 & x \leq 0 \\ u_2 & x > 0 \end{cases}$ with $u_1 > u_2$.

The Rankine-Hugoniot condition can be written:

$$s = \frac{[u]}{[u^2]} = \frac{u_1 - u_2}{u_1^2 - u_2^2} = \frac{1}{u_1 + u_2}$$

Therefore, the shockwave solution of the Riemann problem is

$$u(x, t) = \begin{cases} u_1 & x \leq s t \\ u_2 & x > s t \end{cases}$$

Definition (Entropy Condition)
 The wave also satisfies:

$$s < \frac{f'(u_1)}{u_1} < \frac{f'(u_2)}{u_2}$$

with discontinuity initial condition $u(x, 0) = \begin{cases} u_1 & x \leq 0 \\ u_2 & x > 0 \end{cases}$

A valid Rankine-Hugoniot condition for the wave solution is:

$$s < \frac{f'(u_1)}{u_1} < \frac{f'(u_2)}{u_2}$$

Problem (Integral Solution)
 Let the Riemann problem with Burgers' equation $u_t + u u_x = 0$ in $\mathbb{R} \times (0, \infty)$ and $u(x, 0) = u_0(x)$ for given. Let

$$u_0(x) = \begin{cases} u_1 & x \leq 0 \\ u_2 & x > 0 \end{cases}$$

with $u_1 < u_2$.

Then the Rankine-Hugoniot condition is given by

$$s = \frac{[u]}{[u^2]} = \frac{u_1 - u_2}{u_1^2 - u_2^2} = \frac{1}{u_1 + u_2}$$

and

$$s < \frac{f'(u_1)}{u_1} < \frac{f'(u_2)}{u_2}$$

is the Rankine-Hugoniot condition for the wave solution.

Differential Equations II



Integral Equations and Convolution

Integral Solution and Entropy Condition

Example (Rankine-Hugoniot Condition for Burgers' Equation)
 Consider Burgers' Equation $u_t + u u_x = 0$ in $\mathbb{R} \times (0, \infty)$ and $u(x, 0) = u_0(x)$. Let

$$u_0(x) = \begin{cases} u_1 & x \leq 0 \\ u_2 & x > 0 \end{cases}$$

with $u_1 > u_2$.

The Rankine-Hugoniot condition can be written:

$$s = \frac{[u]}{[u^2]} = \frac{u_1 - u_2}{u_1^2 - u_2^2} = \frac{1}{u_1 + u_2}$$

Therefore, the shockwave solution of the Riemann problem is

$$u(x, t) = \begin{cases} u_1 & x \leq s t \\ u_2 & x > s t \end{cases}$$

Proposition (Satisfaction of the Entropy Solution)
 If an integral solution satisfies the entropy condition, then this solution is unique.

Entropy condition is unique solution.

1

Question: Which is the physically correct solution?

Ans: An additional condition shows the right integral solution.

Rankine-Hugoniot Condition
 Consider the Riemann problem with Burgers' equation $u_t + u u_x = 0$ in $\mathbb{R} \times (0, \infty)$ and $u(x, 0) = u_0(x)$ for given. Let

$$u_0(x) = \begin{cases} u_1 & x \leq 0 \\ u_2 & x > 0 \end{cases}$$

with $u_1 < u_2$.

Furthermore, let $f \in C^2(\mathbb{R})$ and $f'' > 0$, i.e. the flux function is strictly convex. Set $s = \frac{[f]}{[u]}$.

Remark:

• The assumption $f'' > 0$ in \mathbb{R} holds: $u_1 < u_2 \Rightarrow f'(u_1) < f'(u_2)$.

• There are exactly two types of characteristics:

$$x(t) = u_1 t + x_0 \quad \text{and} \quad x(t) = u_2 t + x_0$$

• The family of curves does not fill the whole of $\mathbb{R} \times \mathbb{R}^+$.

• The integral

$$\int_{\mathbb{R}} u(x, t) dx = \int_{\mathbb{R}} u_0(x) dx$$

is not correct here as arbitrary integral solution could hold.

Proposition (Rankine-Hugoniot)

Let the Riemann problem with Burgers' equation $u_t + u u_x = 0$ in $\mathbb{R} \times (0, \infty)$ and $u(x, 0) = u_0(x)$ for given. Let

$$u_0(x) = \begin{cases} u_1 & x \leq 0 \\ u_2 & x > 0 \end{cases}$$

with $u_1 < u_2$.

Then the Rankine-Hugoniot condition is given by

$$s = \frac{[u]}{[u^2]} = \frac{u_1 - u_2}{u_1^2 - u_2^2} = \frac{1}{u_1 + u_2}$$

an integral solution of the Riemann problem.

Remark: Satisfying the Rankine-Hugoniot condition is a necessary condition.

1

Problem: Integral solution is not unique!

Example: Consider the Riemann problem of Burgers' equation with initial condition

$$u_0(x) = \begin{cases} 1 & x \leq 0 \\ 2 & x > 0 \end{cases}$$

But integral solution hold

$$u(x, t) = \begin{cases} 1 & x \leq t \\ 2 & x > t \end{cases}$$

and

$$u(x, t) = \begin{cases} 1 & x \leq 2t \\ 2 & x > 2t \end{cases}$$

is another wave 2

Differential Equations II



Integral Solutions and Entropy Condition

Recap

Example: (Burger' Equation)
 Burgers' equation is given by the flux function $f(u) = \frac{u^2}{2}$, resp. by the Cauchy problem

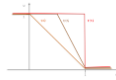
$$u_t + uu_x = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}, \infty[\\ u = u_0 \quad \text{on } \mathbb{R} \times \{t=0\}$$

- The solution is given by $u(t) = u_0 + \text{Rank}(u_0)$.
- If u_0 is given by $u_0(x) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$

then $x(t)$ develops a singularity for $t=1$.

- A classical solution of Burger' equation exists only locally for $0 \leq t < 1$.
- The local solution for $t \in [0, 1[$ is

$$u(x,t) = \begin{cases} 1 & x < 1 \\ \frac{x}{1-t} & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$



Definition: (weak solution)

A function $u \in L^\infty(\mathbb{R} \times [0, \infty[)$ is called **integral solution** or **weak solution** of the conservation law $u_t + f(u)_x = 0$, if for all test functions φ :

$$\int_0^\infty \int_{-\infty}^\infty (u_t + f(u)_x) \varphi dx dt + \int_{-\infty}^\infty u_0(x) \varphi(x,0) dx = 0.$$

Remarks:

- A weak solution **needs not be differentiable** function!
- It even can have a **jump**.

Definition: (Riemann problem)

The initial value problem

$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R} \times]0, \infty[\\ u = u_0 \quad \text{on } \mathbb{R} \times \{t=0\}$$

with discontinuous initial conditions

$$u_0(x) = \begin{cases} u_l & : x \leq 0 \\ u_r & : x > 0 \end{cases}$$

is called **Riemann problem** for the scalar conservation law.

Proposition: (Rankine-Hugoniot condition)

If $x = s(t)$ is a shock front of a shock wave solution of $u_t + f(u)_x = 0$, then for the shock speed $s(t)$ the **Rankine-Hugoniot condition** holds:

$$s \equiv \frac{[f]}{[u]} = \frac{f(u(s(t)^-, t)) - f(u(s(t)^+, t))}{u(s(t)^-, t) - u(s(t)^+, t)} = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Definition: (shock wave solution)

A **shock wave solution** u is a weak solution of the conservation law

$$u_t + f(u)_x = 0$$

if a shock front $x = s(t)$, $s \in C^1$ exists, such that u is a classical solution for each $x < s(t)$ and $x > s(t)$ and it has a jump at $x = s(t)$ with height

$$[u](t) = u(s(t)^+, t) - u(s(t)^-, t) = u_r - u_l.$$

$s(t)$ is called **shock speed**.

Example: (Burgers' Equation)

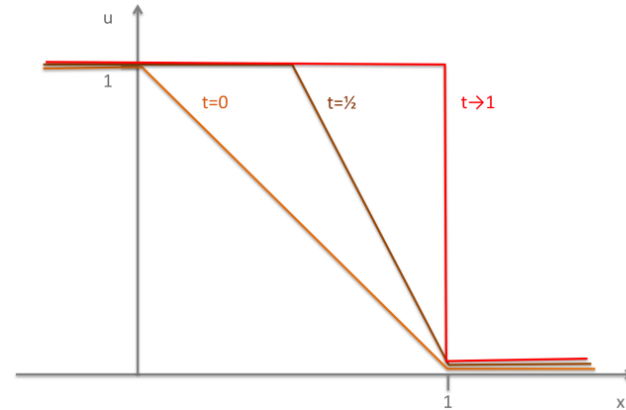
Burgers' Gleichung is given by the flux function $f(u) = \frac{u^2}{2}$, resp. by the Cauchy problem

$$\begin{aligned} u_t + uu_x &= 0 && \text{in } \mathbb{R} \times]0, \infty[\\ u &= u_0 && \text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

- The solution is given by $u(t) = x_0 + tu_0(x_0)$.
- If u_0 is given by

$$u_0(x) = \begin{cases} 1 & : x \leq 0 \\ 1 - x & : 0 < x < 1 \\ 0 & : x \geq 1 \end{cases}$$

then $x(t)$ develops a singularity for $t \rightarrow 1$.



- A classical solution of Burgers' equation exists only **locally** for $0 \leq t < 1$.
- The local solution for $t \in [0, 1[$ is:

$$u(x, t) = \begin{cases} 1 & : x < 1 \\ \frac{(1-x)}{(1-t)} & : 0 \leq t \leq x \leq 1 \\ 0 & : x > 1 \end{cases}$$

Definition: (weak solution)

A function $u \in L^\infty(\mathbb{R} \times [0, \infty[)$ is called **integral solution** or **weak solution** of the conservation law $u_t + f(u)_x = 0$, if for all test functions v :

$$\int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x) dxdt + \int_{-\infty}^\infty u_0(x)v(x, 0) dx = 0.$$

Remarks:

- A weak solution **needs not be** differentiable function!
- It even can have a **jump**.

Definition: (Riemann problem)

The initial value problem

$$\begin{aligned} u_t + f(u)_x &= 0 && \text{in } \mathbb{R} \times]0, \infty[\\ u &= u_0 && \text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

with discontinuous initial conditions

$$u_0(x) = \begin{cases} u_l & : x \leq 0 \\ u_r & : x > 0 \end{cases}$$

is called **Riemann problem** for the scalar conservation law.

Definition: (shock wave solution)

A **shock wave solution** u is a weak solution of the conservation law

$$u_t + f(u)_x = 0$$

if a **shock front** $x = s(t)$, $s \in C^1$ exists, such that u is a classical solution for each $x < s(t)$ and $x > s(t)$ and u has a jump at $x = s(t)$ with height

$$[u](t) = u(s(t)^+, t) - u(s(t)^-, t) = u_r - u_l.$$

$\dot{s}(t)$ is called **shock speed**.

Proposition: (Rankine-Hugoniot condition)

If $x = s(t)$ is a shock front of a shock wave solution of $u_t + f(u)_x = 0$, then for the shock speed $\dot{s}(t)$ the **Rankine-Hugoniot condition** holds:

$$\dot{s} = \frac{[f]}{[u]} = \frac{f(u(s(t)^-, t)) - f(u(s(t)^+, t))}{u(s(t)^-, t) - u(s(t)^+, t)} = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Integral Solution and Entropy Condition

Example (Rankine-Hugoniot Condition for Burgers' Equation)
 Consider Burgers' Equation $u_t + uu_x = 0$ in $\mathbb{R} \times (0, \infty)$ and $u(x, t=0) = u_0$. Let

$$u_0(x) = \begin{cases} u_1 & : x \leq 0 \\ u_2 & : x > 0 \end{cases} \text{ with } u_1 > u_2.$$

The Rankine-Hugoniot condition can be written:

$$k \frac{[u]}{[t]} = \frac{[u^2]}{[u]} = \frac{u_1^2 - u_2^2}{2(u_1 - u_2)} = \frac{1}{2}(u_1 + u_2).$$

Therefore, the shockwave solution of the Riemann problem is

$$u(x, t) = \begin{cases} u_1 & : x \leq \frac{1}{2}(u_1 + u_2)t \\ u_2 & : x > \frac{1}{2}(u_1 + u_2)t \end{cases}$$

Rarefaction Wave:
 Consider the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R} \times [0, \infty)$ and $u(x, t=0) = u_0$. Let

$$u_0(x) = \begin{cases} u_1 & : x \leq 0 \\ u_2 & : x > 0 \end{cases} \text{ with } u_1 < u_2.$$

Furthermore, let $f \in C^1(\mathbb{R})$ and $f'' > 0$, i.e. the flux function is strictly convex. Set $g := (f')^{-1}$.

Remarks:

- By assumption $f'' > 0$, so it holds: $u_1 < u_2 \Rightarrow f'(u_1) < f'(u_2)$.
- There are exactly two types of characteristics:
 $x(t) = u_1 + f'(u_1)t$ and $x(t) = u_2 + f'(u_2)t$.
- The family of curves does not fill the whole of $\mathbb{R} \times \mathbb{R}^+$.
- The area of
 $\Omega = \{(x, t) \in \mathbb{R} \times \mathbb{R}^+ : f'(u_1)t < x < f'(u_2)t\}$
 is not covered, here an arbitrary integral solution could hold!

Proposition (Uniqueness of the Entropy Solution)
 If an integral solution satisfies the entropy condition, then this solution is unique.
(Entropy condition is not satisfied)

Question: Which is the physically correct solution?
Idea: An additional condition chooses the right integral solution.

Proposition (Rarefaction Wave)
 Let the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R} \times [0, \infty)$ and $u(x, t=0) = u_0$ be given. Let

$$u_0(x) = \begin{cases} u_1 & : x \leq 0 \\ u_2 & : x > 0 \end{cases} \text{ with } u_1 < u_2.$$

Then the rarefaction wave is given by

$$u(x, t) = \begin{cases} u_1 & : x < f'(u_1)t \\ g\left(\frac{x}{t}\right) & : f'(u_1)t \leq x \leq f'(u_2)t \\ u_2 & : x > f'(u_2)t \end{cases}$$

an integral solution of the Riemann problem.
Remark: Specifically, the rarefaction wave is a continuous function.

Problem: Integral solutions are not unique!
Example: Consider the Riemann problem of Burgers' equation with initial conditions

$$u_0(x) = \begin{cases} 0 & : x \leq 0 \\ 1 & : x > 0 \end{cases}$$

Both integral solutions hold

$$u_1(x, t) = \begin{cases} 0 & : x \leq 0 \\ 1 & : x > 0 \end{cases} \text{ (shock wave)}$$

and

$$u_2(x, t) = \begin{cases} 0 & : x < 0 \\ \frac{x}{t} & : 0 \leq x \leq t \\ 1 & : x > t \end{cases} \text{ (rarefaction wave)}$$

Example: (Rankine-Hugoniot Condition for Burgers' Equation)

Consider Burgers' Equation $u_t + uu_x = 0$ in $\mathbb{R} \times]0, \infty[$ and $u(x, t = 0) = x_0$. Let

$$u_0(x) = \begin{cases} u_l & : x \leq 0 \\ u_r & : x > 0 \end{cases} \quad \text{with } u_l > u_r.$$

The Rankine-Hugoniot condition can be written:

$$\dot{s} = \frac{[f]}{[u]} = \frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_r} = \frac{(u_l + u_r)(u_l - u_r)}{2(u_l - u_r)} = \frac{1}{2}(u_l + u_r).$$

Therefore, the shockwave solution of the Riemann problem is

$$u(x, t) = \begin{cases} u_l & : x \leq \frac{1}{2}(u_l + u_r)t \\ u_r & : x > \frac{1}{2}(u_l + u_r)t \end{cases}$$

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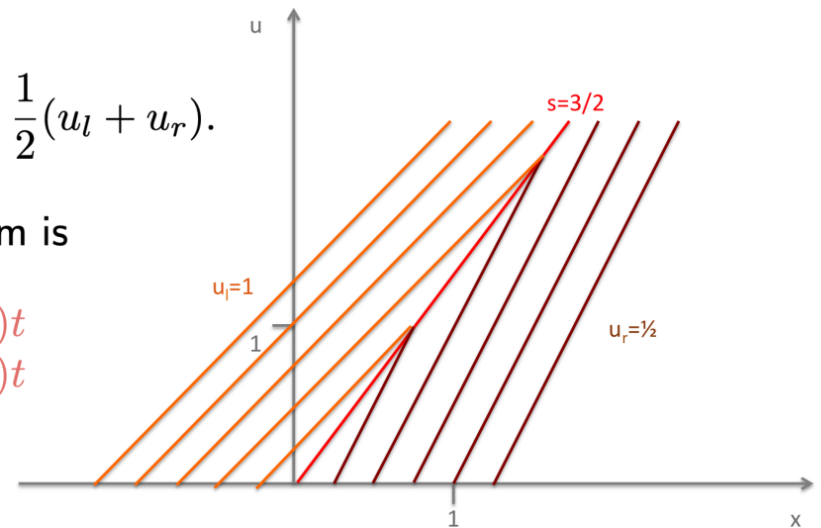
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Therefore, the shockwave solution of the Riemann problem is

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Rarefaction Wave:

Consider the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R} \times]0, \infty[$ and $u(x, t = 0) = x_0$. Let

$$u_0(x) = \begin{cases} u_l & : x \leq 0 \\ u_r & : x > 0 \end{cases} \quad \text{with } u_l < u_r.$$

Furthermore, let $f \in \mathcal{C}^2(\mathbb{R})$ and $f'' > 0$, i.e. the flux function is *strictly convex*. Set $g := (f')^{-1}$.

Remarks:

- By assumption $f' > 0$, so it holds: $u_l < u_r \Rightarrow f'(u_l) < f'(u_r)$.
- There are exactly two types of characteristics:

$$x(t) = x_0 + f'(u_l)t \quad \text{and} \quad x(t) = x_0 + f'(u_r)t.$$

- The family of curves does not fill the whole of $\mathbb{R} \times \mathbb{R}^+$!
- The area of

$$\Omega = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : f'(u_l)t < x < f'(u_r)t\}$$

is not covered; here an arbitrary integral solution could hold!

Remarks:

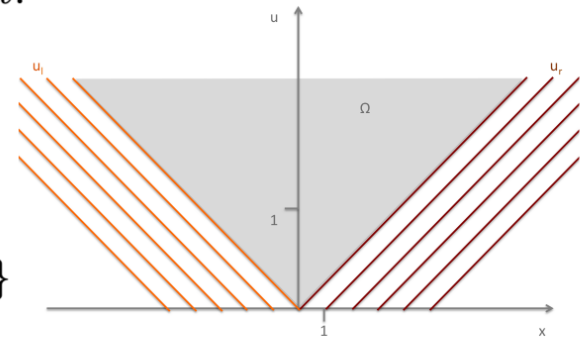
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- The area of

$$\Omega = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : f'(u_l)t < x < f'(u_r)t\}$$

is not covered; here an arbitrary integral solution could hold!



Proposition: (Rarefaction Wave)

Let the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R} \times]0, \infty[$ and $u(x, t = 0) = x_0$ be given. Let

$$u_0(x) = \begin{cases} u_l & : x \leq 0 \\ u_r & : x > 0 \end{cases} \quad \text{with } u_l < u_r.$$

Then the rarefaction wave is given by

$$u(x, t) = \begin{cases} u_l & : x < f'(u_l)t \\ g\left(\frac{x}{t}\right) & : f'(u_l)t \leq x \leq f'(u_r)t \\ u_r & : x > f'(u_r)t \end{cases}$$

an integral solution of the Riemann problem.

Remark: Specifically, the rarefaction wave is a continuous function.

1

Problem: Integral solutions are not unique!

Example: Consider the Riemann problem of Burgers' equation with initial conditions

$$u_0(x) = \begin{cases} 0 & : x \leq 0 \\ 1 & : x > 0 \end{cases}$$

Both integral solutions hold

$$u_1(x, t) = \begin{cases} 0 & : x \leq \frac{t}{2} \\ 1 & : x > \frac{t}{2} \end{cases} \quad (\text{shock wave})$$

and

$$u_2(x, t) = \begin{cases} 0 & : x < 0 \\ \frac{x}{t} & : 0 \leq x \leq t \\ 1 & : x > t \end{cases} \quad (\text{rarefaction wave})$$

2

Question: Which is the physically correct solution?

Idea: An additional condition chooses the right integral solution.

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Idea: An additional condition chooses the right integral solution.

Definition: (Entropy Condition)

An integral solution is called **entropy solution**, if the solution fulfills the **entropy condition** or **Lax-Oleinik condition**:

There exists $C > 0$, such that for all $x, z \in \mathbb{R}$, $t > 0$ with $z > 0$ it holds:

$$u(t, x + z) - u(t, x) < \frac{C}{t}z.$$

Remark: Named after Olga A. Oleinik (1925-2001) and Peter D. Lax (*1926)



<https://mathshistory.st-andrews.ac.uk/Biographies/Oleinik/pictdisplay/>



<https://abelprize.no>

Proposition: (Uniqueness of the Entropy Solution)

If an integral solution satisfies the entropy condition, then this solution is unique.

Entropy solutions are unique solutions.

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Recap

Definition (Riemann Integral)
 Let $f: [a, b] \rightarrow \mathbb{R}$ be a function. A partition P of $[a, b]$ is a finite set of points $a = x_0 < x_1 < \dots < x_n = b$. The Riemann sum of f over P is
$$S(f, P) = \sum_{i=1}^n f(\xi_i) (x_i - x_{i-1})$$
 where $\xi_i \in [x_{i-1}, x_i]$. The function f is Riemann integrable on $[a, b]$ if
$$\lim_{\|P\| \rightarrow 0} S(f, P) = I$$
 exists and is independent of the choice of ξ_i . In this case, $I = \int_a^b f(x) dx$.

Theorem (Riemann Integrability Criterion)
 A function $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for any partition P with $\|P\| < \delta$, $|S(f, P) - I| < \epsilon$.

Example
 Let $f(x) = x^2$ on $[0, 1]$. Then $\int_0^1 x^2 dx = \frac{1}{3}$.



Integral Solution and Entropy Condition

Definition (Riemann Integral)
 Consider the Riemann problem with Burgers' equation $u_t + u u_x = 0$ in $(\mathbb{R}^+) \times \mathbb{R}$ and $u(x, 0) = u_0(x)$. Let $u_0(x) = \begin{cases} u_1 & x \leq 0 \\ u_2 & x > 0 \end{cases}$ with $u_1 < u_2$. Further, let $f \in C^1(\mathbb{R})$ and $f'' > 0$, i.e. the flux function is strictly convex. Set $p = (f')^{-1}$.

Proposition (Riemann Solution)
 The Riemann solution is
$$u(x, t) = \begin{cases} u_1 & x \leq 0 \\ p\left(\frac{x}{t}\right) & 0 < x < u_1 t \\ u_2 & x \geq u_2 t \end{cases}$$

Proposition (Entropy Condition)
 If an integral solution satisfies the entropy condition, then this solution is unique.

Question: Which is the physically correct solution?
Answer: An additional condition chooses the right integral solution.

Definition (Entropy Condition)
 An integral solution is called entropy satisfying if the entropy flux is non-increasing across the shock.

Proposition (Riemann Solution)
 Let the Riemann problem with Burgers' equation $u_t + u u_x = 0$ in $(\mathbb{R}^+) \times \mathbb{R}$ and $u(x, 0) = u_0(x)$ be given. Let $u_0(x) = \begin{cases} u_1 & x \leq 0 \\ u_2 & x > 0 \end{cases}$ with $u_1 < u_2$. Then the entropy condition is given by
$$u_1 < p\left(\frac{x}{t}\right) < u_2$$
 for $0 < x < u_1 t$.

Problem: Integral solutions are not unique!
Example: Consider the Riemann problem of Burgers' equation with initial condition $u_0(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$. But integral solutions hold
$$u(x, t) = \begin{cases} 1 & x \leq 0 \\ \frac{x}{t} & 0 < x < t \\ 0 & x \geq t \end{cases}$$
 (shock wave) and
$$u(x, t) = \begin{cases} 1 & x \leq 0 \\ 0 & 0 < x < t \\ 1 & x \geq t \end{cases}$$
 (shock wave 2).