Recap

Definition: (Riemann problem)
The initial value problem

$$\begin{aligned} u_t + f(u)_x &= 0 \quad \text{in } \mathbb{R}\times]0,\infty[\\ u &= u_0 \quad \text{on } \mathbb{R}\times\{t=0\} \end{aligned}$$

with discontinuous initial conditions

$$u_0(x) = \begin{cases} u_l & : \; x \le 0 \\ u_r & : \; x > 0 \end{cases}$$

is called Riemann problem for the scalar conservation law.

Differential Equations II

Integral Solutions and Entropy Condition

Integral Solution and Entropy Condition

Example: (Rankine-Hugoniot Condition for Burgers' Equation)
Consider Burgers' Equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x,t=0) = x_0$. Let

$$u_0(x) = \begin{cases} u_l & : \; x \le 0 \\ u_r & : \; x > 0 \end{cases} \quad \text{with } u_l > u_r.$$

The Rankine-Hugoniot condition can be written:

$$\dot{s} = \frac{[f]}{[u]} = \frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_r} = \frac{(u_l+u_r)(u_l-u_r)}{2(u_l-u_r)} = \frac{1}{2}(u_l+u_r).$$

Therefore, the shockwave solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_l & : \; x \le \frac{1}{2}(u_l+u_r)t \\ u_r & : \; x > \frac{1}{2}(u_l+u_r)t \end{cases}$$

Rarefaction Wave:
Consider the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R}\times[0,\infty[$ and $u(x,t=0) = x_0$. Let

$$u_0(x) = \begin{cases} u_l & : \; x \le 0 \\ u_r & : \; x > 0 \end{cases} \quad \text{with } u_l < u_r.$$

Furthermore, let $f \in C^2(\mathbb{R})$ and $f'' > 0$, i.e. the flux function is *strictly convex*. Set $g := (f')^{-1}$.

Remarks:

- By assumption $f'' > 0$, so it holds: $u_l < u_r \Rightarrow f'(u_l) < f'(u_r)$.
- There are exactly two types of characteristics:
$$x(t) = x_0 + f'(u_l)t \quad \text{and} \quad x(t) = x_0 + f'(u_r)t.$$
- The family of curves does not fill the whole of $\mathbb{R}\times\mathbb{R}^+$!
- The area of
$$\Omega = \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ : f'(u_l)t < x < f'(u_r)t\}$$
is not covered; here an arbitrary integral solution could hold!

Proposition: (Uniqueness of the Entropy Solution)
If an integral solution satisfies the entropy condition, then this solution is unique.
Entropy solutions are unique solutions.

3

Question: Which is the physically correct solution?

Idea: An additional condition choses the right integral solution.

Proposition: (Rarefaction Wave)
Let the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty]$ and $u(x,t=0) = x_0$ be given. Let

$$u_0(x) = \begin{cases} u_l & : \; x \le 0 \\ u_r & : \; x > 0 \end{cases} \quad \text{with } u_l < u_r.$$

Then the rarefaction wave is given by

$$u(x,t) = \begin{cases} u_l & : \; x < f'(u_l)t \\ g(\frac{x}{t}) & : \; f'(u_l)t \le x \le f'(u_r)t \\ u_r & : \; x > f'(u_r)t \end{cases}$$

an integral solution of the Riemann problem.
Remark: Specifically, the rarefaction wave is a continuous function.

1

Problem: Integral solutions are not unique!
Example: Consider the Riemann problem of Burgers' equation with initial conditions

$$u_0(x) = \begin{cases} 0 & : \; x \le 0 \\ 1 & : \; x > 0 \end{cases}$$

Both integral solutions hold

$$u_1(x,t) = \begin{cases} 0 & : \; x \le \frac{t}{2} \\ 1 & : \; x > \frac{t}{2} \end{cases} \quad \text{(shock wave)}$$

and

$$u_2(x,t) = \begin{cases} 0 & : \; x < 0 \\ \frac{x}{t} & : \; 0 \le x \le t \\ 1 & : \; x > t \end{cases} \quad \text{(rarefaction wave)}$$

2

Differential Equations II

Integral Solutions and Entropy Condition

Recap

Example: (Burgers' Equation)
Burgers' Gleichung is given by the flux function $f(u) = \frac{u^2}{2}$, resp. by the Cauchy problem

$$\begin{aligned} u_t + uu_x &= 0 \quad \text{in } \mathbb{R}\times]0,\infty[\\ u &= u_0 \quad \text{on } \mathbb{R}\times\{t=0\} \end{aligned}$$

- The solution is given by $u(t) = x_0 + tu_0(x_0)$.
- If u_0 is given by
$$u_0(x) = \begin{cases} 1 & : x \le 0 \\ 1-x & : 0 < x < 1 \\ 0 & : x \ge 1 \end{cases}$$
then $x(t)$ develops a singularity for $t \to 1$.
- A classical solution of Burgers' equation exists only locally for $0 \le t < 1$.
- The local solution for $t \in [0,1[$ is:
$$u(x,t) = \begin{cases} 1 & : x < 1 \\ \frac{(1-x)}{(1-t)} & : 0 \le t \le x \le 1 \\ 0 & : x > 1 \end{cases}$$

Definition: (weak solution)
A function $u \in L^\infty(\mathbb{R}\times[0,\infty[)$ is called integral solution or weak solution of the conservation law $u_t + f(u)_x = 0$, if for all test functions v:

$$\int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x)\ dxdt + \int_{-\infty}^\infty u_0(x)v(x,0)\ dx = 0.$$

Remarks:

- A weak solution needs not be differentiable function!
- It even can have a jump.

Definition: (Riemann problem)
The initial value problem

$$\begin{aligned} u_t + f(u)_x &= 0 \quad \text{in } \mathbb{R}\times]0,\infty[\\ u &= u_0 \quad \text{on } \mathbb{R}\times\{t=0\} \end{aligned}$$

with discontinuous initial conditions

$$u_0(x) = \begin{cases} u_l & : x \le 0 \\ u_r & : x > 0 \end{cases}$$

is called Riemann problem for the scalar conservation law.

Proposition: (Rankine-Hugoniot condition)
If $x = s(t)$ is a shock front of a shock wave solution of $u_t + f(u)_x = 0$, then for the shock speed $\dot{s}(t)$ the Rankine-Hugoniot condition holds:

$$\dot{s} = \frac{[f]}{[u]} = \frac{f(u(s(t)^-,t)) - f(u(s(t)^+,t))}{u(s(t)^-,t) - u(s(t)^+,t)} = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Definition: (shock wave solution)
A shock wave solution u is a weak solution of the conservation law

$$u_t + f(u)_x = 0$$

if a shock front $x = s(t)$, $s \in C^1$ exists, such that u is a classical solution for each $x < s(t)$ and $x > s(t)$ and u has a jump at $x = s(t)$ with height

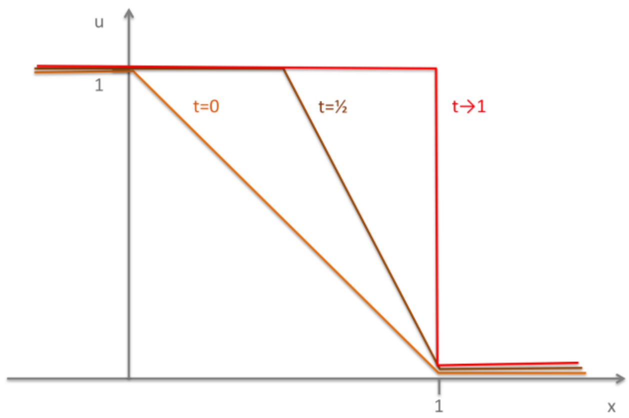
$$[u](t) = u(s(t)^+,t) - u(s(t)^-,t) = u_r - u_l.$$

$\dot{s}(t)$ is called shock speed.

Example: (Burgers' Equation)

Burgers' Gleichung is given by the flux function $f(u) = \frac{u^2}{2}$, resp. by the Cauchy problem

$$\begin{aligned} u_t + uu_x = &\ 0 \quad \text{in } \mathbb{R}\times]0,\infty[\\ u = &\ u_0 \quad \text{on } \mathbb{R}\times\{t=0\} \end{aligned}$$



- The solution is given by $u(t) = x_0 + tu_0(x_0)$.
- If u_0 is given by

$$u_0(x) = \begin{cases} 1 & : \quad x \leq 0 \\ 1-x & : \quad 0 < x < 1 \\ 0 & : \quad x \geq 1 \end{cases}$$

  then $x(t)$ develops a singularity for $t \to 1$.
- A classical solution of Burgers' equation exists only locally for $0 \leq t < 1$.
- The local solution for $t \in [0,1[$ is:

$$u(x,t) = \begin{cases} 1 & : \quad x < 1 \\ \frac{(1-x)}{(1-t)} & : \quad 0 \leq t \leq x \leq 1 \\ 0 & : \quad x > 1 \end{cases}$$

Definition: (weak solution)
A function $u \in L^\infty(\mathbb{R} \times [0, \infty[)$ is called integral solution or weak solution of the conservation law $u_t + f(u)_x = 0$, if for all test functions v:

$$\int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x)\ dxdt + \int_{-\infty}^\infty u_0(x)v(x,0)\ dx = 0.$$

Remarks:

- A weak solution needs not be differentiable function!
- It even can have a jump.

Definition: (Riemann problem)
The initial value problem

$$\begin{aligned} u_t + f(u)_x = & \quad 0 & \text{in } \mathbb{R}\times]0,\infty[\\ u = & \quad u_0 & \text{on } \mathbb{R}\times\{t=0\} \end{aligned}$$

with discontinuous initial conditions

$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases}$$

is called Riemann problem for the scalar conservation law.

Definition: (shock wave solution)
A shock wave solution u is a weak solution of the conservation law

$$u_t + f(u)_x = 0$$

if a shock front $x = s(t)$, $s \in C^1$ exists, such that u is a classical solution for each $x < s(t)$ and $x > s(t)$ and u has a jump at $x = s(t)$ with height

$$[u](t) = u(s(t)^+, t) - u(s(t)^-, t) = u_r - u_l.$$

$\dot{s}(t)$ is called shock speed.

Proposition: (Rankine-Hugoniot condition)
If $x = s(t)$ is a shock front of a shock wave solution of $u_t + f(u)_x = 0$, then for the shock speed $\dot{s}(t)$ the Rankine-Hugoniot condition holds:

$$\dot{s} = \frac{[f]}{[u]} = \frac{f(u(s(t)^-, t)) - f(u(s(t)^+, t))}{u(s(t)^-, t) - u(s(t)^+, t)} = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

ations II

py Condition

Integral Solution and Entropy Condition

Example: (Rankine-Hugoniot Condition for Burgers' Equation)
Consider Burgers' Equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x,t=0) = x_0$. Let

$$u_0(x) = \begin{cases} u_l & : \quad x \le 0 \\ u_r & : \quad x > 0 \end{cases} \quad \text{with } u_l > u_r.$$

The Rankine-Hugoniot condition can be written:

$$\dot{s} = \frac{[f]}{[u]} = \frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_r} = \frac{(u_l+u_r)(u_l-u_r)}{2(u_l-u_r)} = \frac{1}{2}(u_l+u_r).$$

Therefore, the shockwave solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_l & : \quad x \le \frac{1}{2}(u_l+u_r)t \\ u_r & : \quad x > \frac{1}{2}(u_l+u_r)t \end{cases}$$

Rarefaction Wave:
Consider the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x,t=0) = x_0$. Let

$$u_0(x) = \begin{cases} u_l & : \quad x \le 0 \\ u_r & : \quad x > 0 \end{cases} \quad \text{with } u_l < u_r.$$

Furthermore, let $f \in C^2(\mathbb{R})$ and $f'' > 0$, i.e. the flux function is *strictly convex*. Set $g := (f')^{-1}$.

Remarks:

- By assumption $f'' > 0$, so it holds: $u_l < u_r \Rightarrow f'(u_l) < f'(u_r)$.
- There are exactly two types of characteristics:
  $$x(t) = x_0 + f'(u_l)t \quad \text{and} \quad x(t) = x_0 + f'(u_r)t.$$
- The family of curves does not fill the whole of $\mathbb{R}\times\mathbb{R}^+$!
- The area of
  $$\Omega = \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ : f'(u_l)t < x < f'(u_r)t\}$$
  is not covered; here an arbitrary integral solution could hold!

Proposition: (Uniqueness of the Entropy Solution)
If an integral solution satisfies the entropy condition, then this solution is unique.
Entropy solutions are unique solutions.

3

Question: Which is the physically correct solution?

Idea: An additional condition choses the right integral solution.

Proposition: (Rarefaction Wave)
Let the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x,t=0) = x_0$ be given. Let

$$u_0(x) = \begin{cases} u_l & : \quad x \le 0 \\ u_r & : \quad x > 0 \end{cases} \quad \text{with } u_l < u_r.$$

Then the rarefaction wave is given by

$$u(x,t) = \begin{cases} u_l & : \quad x < f'(u_l)t \\ g(\frac{x}{t}) & : \quad f'(u_l)t \le x \le f'(u_r)t \\ u_r & : \quad x > f'(u_r)t \end{cases}$$

an integral solution of the Riemann problem.
Remark: Specifically, the rarefaction wave is a continuous function.

1

Problem: Integral solutions are not unique!
Example: Consider the Riemann problem of Burgers' equation with initial conditions

$$u_0(x) = \begin{cases} 0 & : \quad x \le 0 \\ 1 & : \quad x > 0 \end{cases}$$

Both integral solutions hold

$$u_1(x,t) = \begin{cases} 0 & : \quad x \le \frac{t}{2} \\ 1 & : \quad x > \frac{t}{2} \end{cases} \quad \text{(shock wave)}$$

and

$$u_2(x,t) = \begin{cases} 0 & : \quad x < 0 \\ \frac{x}{t} & : \quad 0 \le x \le t \\ 1 & : \quad x > t \end{cases} \quad \text{(rarefaction wave)}$$

2

Example: (Rankine-Hugoniot Condition for Burgers' Equation)
Consider Burgers' Equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x,t=0) = x_0$. Let

$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases} \qquad \text{with } u_l > u_r.$$

The Rankine-Hugoniot condition can be written:

$$\dot{s} = \frac{[f]}{[u]} = \frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_r} = \frac{(u_l + u_r)(u_l - u_r)}{2(u_l - u_r)} = \frac{1}{2}(u_l + u_r).$$

Therefore, the shockwave solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_l & : \quad x \leq \frac{1}{2}(u_l + u_r)t \\ u_r & : \quad x > \frac{1}{2}(u_l + u_r)t \end{cases}$$

Example: (Rankine-Hugoniot Condition for Burgers' Equation)
Consider Burgers' Equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x, t = 0) = x_0$. Let

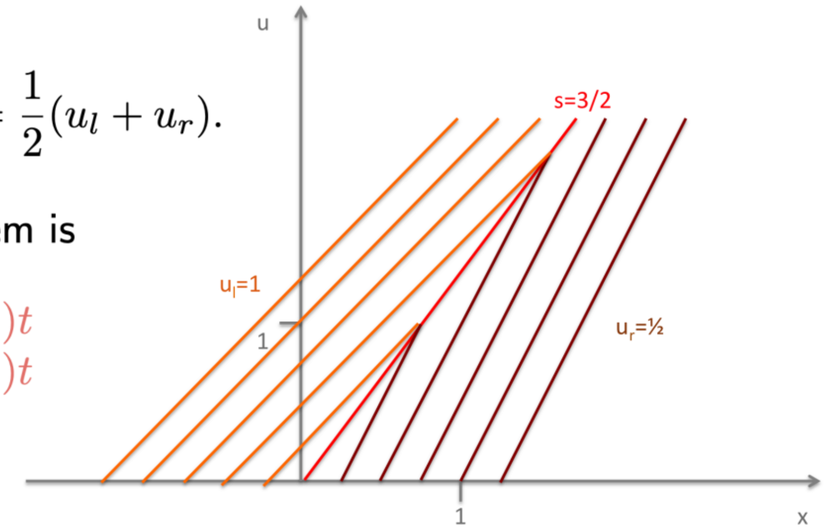
$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases} \qquad \text{with } u_l > u_r.$$

The Rankine-Hugoniot condition can be written:

$$\dot{s} = \frac{[f]}{[u]} = \frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_r} = \frac{(u_l + u_r)(u_l - u_r)}{2(u_l - u_r)} = \frac{1}{2}(u_l + u_r).$$

Therefore, the shockwave solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_l & : \quad x \leq \frac{1}{2}(u_l + u_r)t \\ u_r & : \quad x > \frac{1}{2}(u_l + u_r)t \end{cases}$$



Rarefaction Wave:
Consider the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x, t = 0) = x_0$. Let

$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases} \qquad \text{with } u_l < u_r.$$

Furthermore, let $f \in \mathcal{C}^2(\mathbb{R})$ and $f'' > 0$, i.e. the flux function is *strictly convex*. Set $g := (f')^{-1}$.

Remarks:

- By assumption $f' > 0$, so it holds: $u_l < u_r \;\Rightarrow\; f'(u_l) < f'(u_r)$.
- There are exactly two types of characteristics:

$$x(t) = x_0 + f'(u_l)t \quad \text{and} \quad x(t) = x_0 + f'(u_r)t.$$

- The family of curves does not fill the whole of $\mathbb{R} \times \mathbb{R}^+$!
- The area of

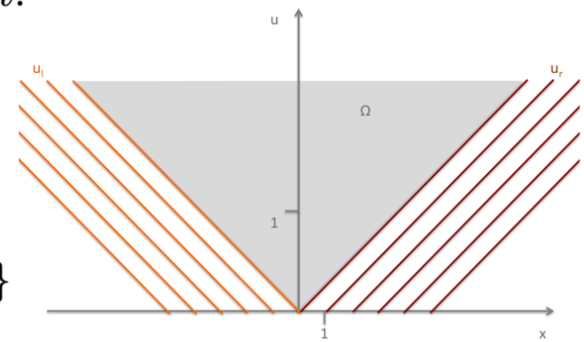
$$\Omega = \{(x.t) \in \mathbb{R} \times \mathbb{R}^+ : f'(u_l)t < x < f'(u_r)t\}$$

is not covered; here an arbitrary integral solution could hold!

Remarks:

- By assumption $f' > 0$, so it holds: $u_l < u_r \;\Rightarrow\; f'(u_l) < f'(u_r)$.
- There are exactly two types of characteristics:

$$x(t) = x_0 + f'(u_l)t \quad \text{and} \quad x(t) = x_0 + f'(u_r)t.$$



- The family of curves does not fill the whole of $\mathbb{R} \times \mathbb{R}^+$!
- The area of

$$\Omega = \{(x.t) \in \mathbb{R} \times \mathbb{R}^+ : f'(u_l)t < x < f'(u_r)t\}$$

is not covered; here an arbitrary integral solution could hold!

Proposition: (Rarefaction Wave)
Let the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x, t = 0) = x_0$ be given. Let

$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases} \qquad \text{with } u_l < u_r.$$

Then the rarefaction wave is given by

$$u(x,t) = \begin{cases} u_l & : \quad x < f'(u_l)t \\ g(\frac{x}{t}) & : \quad f'(u_l)t \leq x \leq f'(u_r)t \\ u_r & : \quad x > f'(u_r)t \end{cases}$$

an integral solution of the Riemann problem.
Remark: Specifically, the rarefaction wave is a continuous function.

1

Problem: Integral solutions are not unique!

Example: Consider the Riemann problem of Burgers' equation with initial conditions

$$u_0(x) = \begin{cases} 0 & : \quad x \leq 0 \\ 1 & : \quad x > 0 \end{cases}$$

Both integral solutions hold

$$u_1(x,t) = \begin{cases} 0 & : \quad x \leq \frac{t}{2} \\ 1 & : \quad x > \frac{t}{2} \end{cases} \qquad \text{(shock wave)}$$

and

$$u_2(x,t) = \begin{cases} 0 & : \quad x < 0 \\ \frac{x}{t} & : \quad 0 \leq x \leq t \\ 1 & : \quad x > t \end{cases} \qquad \text{(rarefaction wave)}$$

2

Question: Which is the physically correct solution?

Idea: An additional condition choses the right integral solution.

Question: Which is the physically correct solution?

Idea: An additional condition choses the right integral solution.

Definition: (Entropy Condition)
An integral solution is called entropy solution, if the solution fulfills the entropy condition or Lax-Oleinik condition:
There exists $C > 0$, such that for all $x, z \in \mathbb{R}$, $t > 0$ with $z > 0$ it holds:

$$u(t, x+z) - u(t, x) < \frac{C}{t} z.$$

Remark: Named after Olga A. Oleinik (1925-2001) and Peter D. Lax (*1926)

https://mathshistory.st-andrews.ac.uk/Biographies/Oleinik/pictdisplay/

https://abelprize.no

Proposition: (Uniqueness of the Entropy Solution)
If an integral solution satisfies the entropy condition, then this solution is unique.
Entropy solutions are unique solutions.

3

Recap

Differential Equations II

Integral Solutions and Entropy Condition

Integral Solution and Entropy Condition

Example: (Rankine-Hugoniot Condition for Burgers' Equation)
Consider Burgers' Equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x,t=0) = x_0$. Let

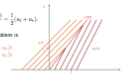
$$u_0(x) = \begin{cases} u_l & : & x \le 0 \\ u_r & : & x > 0 \end{cases} \quad \text{with } u_l > u_r.$$

The Rankine-Hugoniot condition can be written:

$$s = \frac{[f]}{[u]} = \frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_r} = \frac{(u_l+u_r)(u_l-u_r)}{2(u_l-u_r)} = \frac{1}{2}(u_l+u_r).$$

Therefore, the shockwave solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_l & : & x \le \frac{1}{2}(u_l+u_r)t \\ u_r & : & x > \frac{1}{2}(u_l+u_r)t \end{cases}$$



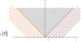
Rarefaction Wave:
Consider the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R}\times[0,\infty[$ and $u(x,t=0) = x_0$. Let

$$u_0(x) = \begin{cases} u_l & : & x \le 0 \\ u_r & : & x > 0 \end{cases} \quad \text{with } u_l < u_r.$$

Furthermore, let $f \in C^2(\mathbb{R})$ and $f'' > 0$, i.e. the flux function is *strictly convex*. Set $g := (f')^{-1}$.

Remarks:

- By assumption $f'' > 0$, so it holds: $u_l < u_r \Rightarrow f'(u_l) < f'(u_r)$.
- There are exactly two types of characteristics:
  $$x(t) = x_0 + f'(u_l)t \quad \text{and} \quad x(t) = x_0 + f'(u_r)t.$$
- The family of curves does not fill the whole of $\mathbb{R}\times\mathbb{R}^+$!
- The area of
  $$\Omega = \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ : f'(u_l)t < x < f'(u_r)t\}$$
  is not covered; here an arbitrary integral solution could hold!



Proposition: (Uniqueness of the Entropy Solution)
If an integral solution satisfies the entropy condition, then this solution is unique.
Entropy solutions are unique solutions.

3

Question: Which is the physically correct solution?

Idea: An additional condition choses the right integral solution.

Definition: (Entropy Condition)
An integral solution is called entropy solution, if the solution fulfills the entropy condition or Lax-Oleinik condition:
There exists $C > 0$, such that for all $x, z \in \mathbb{R}$, $t > 0$ with $z > 0$ it holds:

$$u(t, x+z) - u(t,x) < \frac{C}{t}z.$$

Remark: Named after Olga A. Oleinik (1925-2001) and Peter D. Lax (*1926)

Proposition: (Rarefaction Wave)
Let the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x,t=0) = x_0$ be given. Let

$$u_0(x) = \begin{cases} u_l & : & x \le 0 \\ u_r & : & x > 0 \end{cases} \quad \text{with } u_l < u_r.$$

Then the rarefaction wave is given by

$$u(x,t) = \begin{cases} u_l & : & x < f'(u_l)t \\ g(\frac{x}{t}) & : & f'(u_l)t \le x \le f'(u_r)t \\ u_r & : & x > f'(u_r)t \end{cases}$$

an integral solution of the Riemann problem.
Remark: Specifically, the rarefaction wave is a continuous function.

1

Problem: Integral solutions are not unique!
Example: Consider the Riemann problem of Burgers' equation with initial conditions

$$u_0(x) = \begin{cases} 0 & : & x \le 0 \\ 1 & : & x > 0 \end{cases}$$

Both integral solutions hold

$$u_1(x,t) = \begin{cases} 0 & : & x \le \frac{t}{2} \\ 1 & : & x > \frac{t}{2} \end{cases} \quad \text{(shock wave)}$$

and

$$u_2(x,t) = \begin{cases} 0 & : & x < 0 \\ \frac{x}{t} & : & 0 \le x \le t \\ 1 & : & x > t \end{cases} \quad \text{(rarefaction wave)}$$

2