

Differential Equations II



Partial Differential Equations of Second Order

Definition

Definition: (PDE of 2nd Order)

A linear partial differential equation of 2nd order in n variables is defined by

$$\sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n b_i u_{x_i} + f u = g.$$

Here the terms a_{ij} , b_i , f , and g are functions of $\mathbf{x} = (x_1, \dots, x_n)^T$.
The first term in the equation is called **main part** of the PDE.

Assume w.l.o.g.:

$$a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}), \quad i, j = 1, \dots, n.$$

Special Case: If $a_{ij} \equiv \text{const.}$ for $i, j = 1, \dots, n$, the PDE can be written in **matrix form**:

$$(\nabla^T A \nabla) u + (\mathbf{b}^T \nabla) u + f u = g,$$

with $A = (a_{ij})_{i,j=1,\dots,n}$ symmetric matrix and $\mathbf{b} = (b_1, \dots, b_n)$.

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with $A = (a_{ij})_{i,j=1,\dots,n}$ symmetric matrix and $\mathbf{b} = (b_1, \dots, b_n)$.

Diagonal Form

Remarks:

- Let the PDE be given in matrix form:

$$(\nabla^T A \nabla)u + (\mathbf{b}^T \nabla)u + fu = g,$$
 with $A = (a_{ij})_{i,j=1,\dots,n}$ symmetric, constant.
- Linear algebra: (principal axis transformation)
Any real, symmetric matrix A can be diagonalized:

$$D = S^{-1}AS,$$
 where S can be chosen orthogonal (i.e. $S^{-1} = S^T$).

Ansatz: (Derivation of Normal Form)

- Use coordinate transformation:

$$\mathbf{x} = S\mathbf{y} \quad \Leftrightarrow \quad \mathbf{y} = S^T \mathbf{x}$$
- Set $\tilde{u}(\mathbf{y}) := u(S\mathbf{y})$.
- With $u(\mathbf{x}) = \tilde{u}(S^T \mathbf{x})$ it follows:

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^n \frac{\partial \tilde{u}}{\partial y_j} \frac{\partial y_j}{\partial x_i}$$
- Due to $\frac{\partial y_j}{\partial x_i} = s_{ji}$ we obtain

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^n s_{ji} \frac{\partial \tilde{u}}{\partial y_j}$$
- This yields

$$\nabla_{\mathbf{x}} u(\mathbf{x}) = S \nabla_{\mathbf{y}} \tilde{u}(S^T \mathbf{x})$$
- Formally: $\nabla_{\mathbf{x}} = S \nabla_{\mathbf{y}}$
- Transpose: $\nabla_{\mathbf{y}}^T = (S \nabla_{\mathbf{x}})^T = \nabla_{\mathbf{x}}^T S^T$.

Summary: If u solves the equation

$$(\nabla^T A \nabla)u + (\mathbf{b}^T \nabla)u + fu = g,$$

then for \tilde{u} we obtain the PDE

$$(\nabla^T S^T A S \nabla) \tilde{u} + (\mathbf{b}^T S \nabla) \tilde{u} + \tilde{f} \tilde{u} = \tilde{g}$$

Definition: (Diagonal Form)

Let the PDE of 2nd order $A = (a_{ij})_{i,j=1,\dots,n}$ constant and symmetric

$$(\nabla^T A \nabla)u + (\mathbf{b}^T \nabla)u + fu = g.$$

Then the corresponding diagonal form of the PDE is given by

$$(\nabla^T D \nabla) \tilde{u} + ((S^T \mathbf{b})^T \nabla) \tilde{u} + \tilde{f} \tilde{u} = \tilde{g}$$

with diagonal matrix $D = S^T A S$ and $S^T S = Id$. Here $\mathbf{b} := \mathbf{b}(S\mathbf{y})$ and $\tilde{f}(S\mathbf{y}) := f(S\mathbf{y})$, $\tilde{g}(S\mathbf{y}) := g(S\mathbf{y})$.

1

Remarks:

- Let the PDE be given in matrix form:

$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g,$$

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Any real, symmetric matrix A can be **diagonalized**:

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Ansatz: (Derivation of Normal Forms)

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$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^n \frac{\partial \tilde{u}}{\partial y_j} \frac{\partial y_j}{\partial x_i}.$$

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$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^n s_{ij} \frac{\partial \tilde{u}}{\partial y_j}.$$

- This yields

$$\nabla_x u(\mathbf{x}) = S \nabla_y \tilde{u}(S^\top \mathbf{x})$$

- Formally: $\nabla_x = S \nabla_y$
- Transpose: $\nabla_x^\top = (S \nabla_y)^\top = \nabla_y^\top S^\top$.

Summary: If u solves the equation

$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g,$$

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$$(\nabla^\top S^\top A S \nabla)\tilde{u} + (\mathbf{b}^\top S \nabla)\tilde{u} + \tilde{f}\tilde{u} = \tilde{g}$$

Definition: (Diagonal Form)

Let the PDE of 2nd order ($A = (a_{ij})_{i,j=1,\dots,n}$ constant and symmetric)

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Then the corresponding **diagonal form** of the PDE is given by

$$(\nabla^\top D \nabla)\tilde{u} + ((S^\top \tilde{\mathbf{b}})^\top \nabla)\tilde{u} + \tilde{f}\tilde{u} = \tilde{g}$$

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1

Classification

Definition: (Classification of Partial Differential Equations of 2nd Order)
Let the PDE of 2nd order ($A = (a_{ij})_{i,j=1,\dots,n}$, constant and symmetric)

$$(\nabla^T A \nabla)u + (b^T \nabla)u + fu = g.$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of matrix A .

1. If $\lambda_i \neq 0$ for all $i = 1, \dots, n$ and if all λ_i have equal sign, the equation is called **elliptic**.
2. If $\lambda_i \neq 0$ for all $i = 1, \dots, n$ and if one eigenvalue has different sign to all other $n - 1$ eigenvalues, the equation is called **hyperbolic**.
3. If $\lambda_k = 0$ for at least one $k \in \{1, \dots, n\}$, the equation is called **parabolic**.

Example: Consider the PDE of 2nd order with two independent variables:

$$a_{11} \frac{\partial^2 u}{\partial x_1^2} + a_{12} \left(\frac{\partial^2 u}{\partial x_2 \partial x_1} + \frac{\partial^2 u}{\partial x_1 \partial x_2} \right) + a_{22} \frac{\partial^2 u}{\partial x_2^2} +$$

$$b_1(x_1, x_2) \frac{\partial u}{\partial x_1} + b_2(x_1, x_2) \frac{\partial u}{\partial x_2} + f(x_1, x_2)u = g(x_1, x_2). \quad (2)$$

The diagonal form is given by

$$\lambda_1 \frac{\partial^2 u}{\partial \xi_1^2} + \lambda_2 \frac{\partial^2 u}{\partial \xi_2^2} + \tilde{b}_1 \frac{\partial u}{\partial \xi_1} + \tilde{b}_2 \frac{\partial u}{\partial \xi_2} + \tilde{f}u = \tilde{g}.$$

Then the differential equation is

1. **elliptic**, if $\lambda_1 \cdot \lambda_2 > 0$;
2. **hyperbolic**, if $\lambda_1 \cdot \lambda_2 < 0$;
3. **parabolic**, if $\lambda_1 \cdot \lambda_2 = 0$.

Remark: (Extension to Non-Constant Coefficients)

The classification by types can be extended to cases with non-constant coefficients, as illustrated by the following example: Let

$$y u_{xx} - u_{xy} - u_{yx} + x u_{yy} = 0.$$

Then the coefficient matrix A is given by

$$A = \begin{bmatrix} y & -1 \\ -1 & x \end{bmatrix}.$$

The discriminant D is $D = 1 - xy$. Thus, the equation is

1. **parabolic** on the hyperbola $xy = 1$,
2. **elliptic** in both convex domains $xy > 1$, and
3. **hyperbolic** in the connected domain $xy < 1$.

2

Question: Why do we classify the equations?

Answer: Because each type shows a characteristic solution behavior!

Definition: (Classification of Partial Differential Equations of 2nd Order)

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$$b_1(x_1, x_2) \frac{\partial u}{\partial x_1} + b_2(x_1, x_2) \frac{\partial u}{\partial x_2} + f(x_1, x_2)u = g(x_1, x_2). \quad (2)$$

The diagonal form is given by

$$\lambda_1 \frac{\partial^2 \tilde{u}}{\partial y_1^2} + \lambda_2 \frac{\partial^2 \tilde{u}}{\partial y_2^2} + \tilde{p}_1 \frac{\partial \tilde{u}}{\partial y_1} + \tilde{p}_2 \frac{\partial \tilde{u}}{\partial y_2} + \tilde{f} \tilde{u} = \tilde{g}.$$

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Normal Forms

Definition: (Normal Forms of Partial Differential Equations of 2nd Order)

1. The normal form of an elliptic PDE in n variables $\mathbf{x} = (x_1, \dots, x_n)^T$ is

$$\Delta u + \sum_{i=1}^n b_i u_{x_i} + f u = g.$$

2. The normal form of a hyperbolic PDE in $n+1$ variables $(\mathbf{x}, t) = (x_1, \dots, x_n, t)^T$ is

$$u_{tt} - \Delta u + \sum_{i=1}^n b_i u_{x_i} + f u = g.$$

3. The normal form of a parabolic PDE in $n+1$ variables $(\mathbf{x}, t) = (x_1, \dots, x_n, t)^T$ is

$$\Delta u + b_0 u_t + \sum_{i=1}^n b_i u_{x_i} + f u = g.$$

Remark: in all cases Δ denotes the Laplace operator w.r.t. \mathbf{x} .

Examples:

1. The elliptic Laplace equation $\Delta u = 0.$
2. The hyperbolic wave equation $u_{tt} - \Delta u = 0.$
3. The parabolic heat equation $u_t = \Delta u.$

Definition: (Normal Forms of Partial Differential Equations of 2nd Order)

1. The **normal form of an elliptic PDE** in n variables $\mathbf{x} = (x_1, \dots, x_n)^\top$ is

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2. The **normal form of a hyperbolic PDE** in $n+1$ variables $(\mathbf{x}, t) = (x_1, \dots, x_n, t)^\top$ is

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Examples:

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$$\Delta u = 0.$$

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3. The parabolic heat equation

$$u_t = \Delta u.$$

Well-Posedness

Definition: (Well-Posed Problem)

A **correctly posed problem** (or **well-posed problem**) consists of

- a partial differential equation, defined on a domain, and
- a set of initial and/or boundary conditions,

such that the following properties are fulfilled:

1. **Existence:** There exists at least one solution, that fulfills all above conditions;
2. **Uniqueness:** The solution is unique;
3. **Stability:** The solution depends cont. on the initial/boundary conditions
i.e. small perturbations in the data yield small perturbations in the solution.

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Example: (Well-Posed Wave Equation)

The initial value problem for the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in } [0, \infty) \times \mathbb{R} \quad (1)$$

$$u = f \quad \text{on } \{t = 0\} \times \mathbb{R} \quad (2)$$

$$u_t = g \quad \text{on } \{t = 0\} \times \mathbb{R} \quad (3)$$

is a well-posed hyperbolic problem.

- The uniquely determined solution is given by the representation by d'Alembert:

$$u(t, x) = \frac{1}{2} \left(f(x - ct) + f(x + ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(z) dz \right).$$

- The solution depends continuously on the data, since

$$\|u - \tilde{u}\|_{\infty} \leq \|f - \tilde{f}\|_{\infty} + \|g - \tilde{g}\|_{\infty}.$$

Example: (Hadamard)

The initial value problem for the PDE

$$u_{xx} + u_{yy} = 0 \quad \text{in } \mathbb{R}^2 \quad (1)$$

$$u = f \quad \text{on } \mathbb{R} \times \{y = 0\} \quad (2)$$

$$u_y = g \quad \text{on } \mathbb{R} \times \{y = 0\} \quad (3)$$

is **not** a well-posed elliptic problem!

Example: (Laplace Equation)

The boundary value problem for the two-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0 \quad \text{in } \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

$$u = g \quad \text{on } \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is a well-posed elliptic problem.

The unique solution is given by Poisson's integral form:

$$u(x, y) = \frac{1 - x^2 - y^2}{2\pi} \int_{|\mathbf{s}|=1} \frac{g(\mathbf{x})}{\|\mathbf{x} - \mathbf{z}\|^2} d\mathbf{s}$$

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- The solution depends continuously on the data, since

$$\|u - \tilde{u}\|_{\infty} \leq \|f - \tilde{f}\|_{\infty} + t \|g - \tilde{g}\|_{\infty}.$$

Example: (Hadamard)

The initial value problem for the PDE

$$u_{xx} + u_{yy} = 0 \quad \text{in } \mathbb{R}^2 \quad (1)$$

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$$u_y = g \quad \text{on } \mathbb{R} \times \{y = 0\} \quad (3)$$

is **not** a well-posed elliptic problem!

Reasoning:

- Set $f(x) = g(x) = 0$, then the unique solution is given by

$$u(x, y) = 0.$$

- On the other hand, if $f_n(x) = 0$ and $g_n(x) = \frac{1}{n} \sin(nx)$, for $n \in \mathbb{N}$, the solution is

$$u_n(x, y) = \frac{1}{n^2} \sin(nx) \sinh(ny)$$

- We have

$$\lim_{n \rightarrow \infty} f_n = f = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} g_n = g = 0$$

- But, since $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sinh(ny) = \infty$ ($y \neq 0$), we have

$$\lim_{n \rightarrow \infty} u_n \neq u,$$

thus the solution does not depend continuously on the data!

Example: (Laplace Equation)

The boundary value problem for the two-dimensional Laplace equation

$$\begin{aligned}u_{xx} + u_{yy} &= 0 && \text{in } \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \\u &= g && \text{on } \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}\end{aligned}$$

is a well-posed elliptic problem.

The unique solution is given by Poisson's integral form:

$$u(x, y) = \frac{1 - x^2 - y^2}{2\pi} \oint_{\|\mathbf{z}\|=1} \frac{g(\mathbf{z})}{\|\mathbf{x} - \mathbf{z}\|^2} ds$$

Classification

Definition: (Existence and Uniqueness Theorem for IVP)

Let I be an interval containing t_0 and let x_0 be a point in \mathbb{R}^n . Suppose that f and F are continuous on $I \times \mathbb{R}^n$ and that f is Lipschitz continuous on $I \times \mathbb{R}^n$. Then there exists a unique solution $x(t)$ to the IVP

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

on the interval I .

Example: Consider the IVP $x'(t) = x(t)^2$, $x(0) = 1$. The function $f(t, x) = x^2$ is continuous and Lipschitz continuous on any bounded interval $I \times \mathbb{R}$. Therefore, there exists a unique solution $x(t)$ on I . The solution is $x(t) = \frac{1}{1-t}$.

Question: Why do we need the Lipschitz condition?

Answer: Without the Lipschitz condition, the solution may not be unique. For example, consider the IVP $x'(t) = \sqrt{|x(t)|}$, $x(0) = 0$. The function $f(t, x) = \sqrt{|x|}$ is continuous but not Lipschitz continuous at $x = 0$. In this case, there are multiple solutions, such as $x(t) = 0$ and $x(t) = \frac{1}{4}t^2$.

Normal Forms

Definition: (Normal Form of a First-Order System of ODEs)

- The normal form of a first-order system of ODEs is $x'(t) = f(t, x(t))$, where f is a vector-valued function.
- The normal form of a second-order system of ODEs is $x''(t) = f(t, x(t), x'(t))$, where f is a vector-valued function.
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- Examples:**
- The damped harmonic oscillator: $x'' + \gamma x' + \omega^2 x = 0$
 - The undamped harmonic oscillator: $x'' + \omega^2 x = 0$
 - The pendulum equation: $x'' + \sin(x) = 0$

Diagonal Form

Definition: (Diagonal Form of a First-Order System of ODEs)

Let A be a constant matrix. The diagonal form of a first-order system of ODEs is $x'(t) = Ax(t)$, where A is a diagonal matrix.

Example: Consider the system $x'(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x(t)$. The matrix A is diagonal, so the system is in diagonal form. The solutions are $x_1(t) = e^t$ and $x_2(t) = e^{2t}$.

Question: Why do we need the diagonal form?

Answer: The diagonal form allows us to solve the system more easily. In this case, we can solve each equation separately, which is much simpler than solving a coupled system of equations.

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A second-order partial differential equation (PDE) of 2nd order in x and y is a function $u(x, y)$ defined by

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y)$$

where the coefficients a, b, c, d, e, f, g are functions of x and y . The function u is called the solution of the PDE.

Example: The Laplace equation $u_{xx} + u_{yy} = 0$ is a second-order PDE of 2nd order in x and y .

Special Case: If a, b, c, d, e, f, g are constants, the PDE can be written as

$$u_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$$

with a, b, c, d, e, f, g constants.

Well-Posedness

Definition: (Well-Posedness of a Problem)

A problem is well-posed if it satisfies the following three conditions:

- Existence: A solution exists.
- Uniqueness: The solution is unique.
- Stability: The solution depends continuously on the data.

Example: The Cauchy problem for the Laplace equation is not well-posed. The solution does not depend continuously on the data.

Question: Why do we need well-posedness?

Answer: Well-posedness ensures that the problem is solvable and that the solution is stable. Without well-posedness, the problem may not have a solution, or the solution may be highly sensitive to small changes in the data.