Differential Equations II

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Partial Differential Equations of Second Order

Definition

Definition: (PDE of 2^{nd} Order)
A linear partial differential equation of 2^{nd} order in n variables is defined by

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + fu = g.$$

Here the terms a_{ij}, b_i, f, and g are functions of $\mathbf{x} = (x_1, \dots, x_n)^\top$.
The first term in the equation is called main part of the PDE.
Assume w.l.o.g:

$$a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}), \quad i, j = 1, \dots, n.$$

Special Case: If $a_{ij} \equiv \text{const.}$ for $i, j = 1, \dots, n$, the PDE can be written in matrix form:

$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g,$$

with $A = (a_{ij})_{i,j=1,\dots,n}$ symmetric matrix and $\mathbf{b} = (b_1, \dots, b_n)$.

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Diagonal Form

Remarks:

- Let the PDE be given in matrix form:

$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g,$$

with $A = (a_{ij})_{i,j=1,\dots,n}$ symmetric, constant.

- Linear algebra: (principal axis transformation)
Any real, symmetric matrix A can be diagonalized:

$$D = S^{-1} A S,$$

where S can be chosen orthogonal (i.e. $S^{-1} = S^\top$).

Ansatz: (Derivation of Normal Forms)

- Use coordinate transformation:

$$\mathbf{x} = S\mathbf{y} \quad \Leftrightarrow \quad \mathbf{y} = S^\top \mathbf{x}$$

- Set $\tilde{u}(\mathbf{y}) := u(S\mathbf{y})$.
- With $u(\mathbf{x}) = \tilde{u}(S^\top \mathbf{x})$ it follows:

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial \tilde{u}}{\partial y_j} \frac{\partial y_j}{\partial x_i}.$$

- Due to $\frac{\partial y_j}{\partial x_i} = s_{ij}$ we obtain

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{n} s_{ij} \frac{\partial \tilde{u}}{\partial y_j}.$$

- This yields

$$\nabla_x u(\mathbf{x}) = S \nabla_y \tilde{u}(S^\top \mathbf{x})$$

- Formally: $\nabla_x = S\nabla_y$
- Transpose: $\nabla_x^\top = (S\nabla_y)^\top = \nabla_y^\top S^\top$.

Summary: If u solves the equation

$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g,$$

then for $\tilde{u}$ we obtain the PDE

$$(\nabla^\top S^\top A S \nabla)\tilde{u} + (\mathbf{b}^\top S \nabla)\tilde{u} + \tilde{f}\tilde{u} = \tilde{g}$$

Definition: (Diagonal Form)
Let the PDE of 2nd order ($A = (a_{ij})_{i,j=1\dots,n}$ constant and symmetric)

$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g.$$

Then the corresponding diagonal form of the PDE is given by

$$(\nabla^\top D \nabla)\tilde{u} + ((S^\top \tilde{\mathbf{b}})^\top \nabla)\tilde{u} + \tilde{f}\tilde{u} = \tilde{g}$$

with diagonal matrix $D = S^\top A S$ and $S^\top S = Id$. Here $\tilde{\mathbf{b}} := \mathbf{b}(S\mathbf{y})$ and $\tilde{f}(\mathbf{y}) := f(S\mathbf{y})$, $\tilde{g}(\mathbf{y}) := g(S\mathbf{y})$.

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1

Classification

Definition: (Classification of Partial Differential Equations of 2nd Order)
Let the PDE of 2nd order ($A = (a_{ij})_{i,j=1\ldots,n}$ constant and symmetric)

$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g.$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of matrix A.

1. If $\lambda_i \neq 0$ for all $i = 1, \ldots, n$ and if all λ_i have equal sign, the equation is called elliptic.
2. If $\lambda_i \neq 0$ for all $i = 1, \ldots, n$ and if one eigenvalue has different sign to all other $n-1$ eigenvalues, the equation is called hyperbolic.
3. If $\lambda_k = 0$ for at least one $k \in \{1, \ldots, n\}$, the equation is called parabolic.

Example: Consider the PDE of 2nd order with two independent variables:

$$a_{11}\frac{\partial^2 u}{\partial x_1^2} + a_{12}\left(\frac{\partial^2 u}{\partial x_2 \partial x_1} + \frac{\partial^2 u}{\partial x_1 \partial x_2}\right) + a_{22}\frac{\partial^2 u}{\partial x_2^2} + \quad (1)$$

$$b_1(x_1,x_2)\frac{\partial u}{\partial x_1} + b_2(x_1,x_2)\frac{\partial u}{\partial x_2} + f(x_1,x_2)u = g(x_1,x_2). \quad (2)$$

The diagonal form is given by

$$\lambda_1\frac{\partial^2 \tilde{u}}{\partial y_1^2} + \lambda_2\frac{\partial^2 \tilde{u}}{\partial y_2^2} + \tilde{p}_1\frac{\partial \tilde{u}}{\partial y_1} + \tilde{p}_2\frac{\partial \tilde{u}}{\partial y_2} + \tilde{f}\tilde{u} = \tilde{g}.$$

Then the differential equation is

1. elliptic, if $\lambda_1 \cdot \lambda_2 > 0$;
2. hyperbolic, if $\lambda_1 \cdot \lambda_2 < 0$;
3. parabolic, if $\lambda_1 \cdot \lambda_2 = 0$.

Remark: (Extension to Non-Constant Coefficients)
The classification by types can be extended to cases with non-constant coefficients, as illustrated by the following example: Let

$$yu_{xx} - u_{xy} - u_{yx} + xu_{yy} = 0.$$

Then the coefficient matrix A is given by

$$A = \begin{bmatrix} y & -1 \\ -1 & x \end{bmatrix}.$$

The discriminant D is $D = 1 - xy$. Thus, the equation is

1. parabolic on the hyperbola $xy = 1$,
2. elliptic in both convex domains $xy > 1$, and
3. hyperbolic in the connected domain $xy < 1$.

2

Question: Why do we classify the equations?
Answer: Because each type shows a characteristic solution behavior!

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$$b_1(x_1, x_2)\frac{\partial u}{\partial x_1} + b_2(x_1, x_2)\frac{\partial u}{\partial x_2} + f(x_1, x_2)u = g(x_1, x_2). \tag{2}$$

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$$\lambda_1 \frac{\partial^2 \tilde{u}}{\partial y_1^2} + \lambda_2 \frac{\partial^2 \tilde{u}}{\partial y_2^2} + \tilde{p}_1 \frac{\partial \tilde{u}}{\partial y_1} + \tilde{p}_2 \frac{\partial \tilde{u}}{\partial y_2} + \tilde{f}\tilde{u} = \tilde{g}.$$

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Normal Forms

Definition: (Normal Forms of Partial Differential Equations of 2nd Order)

1. The normal form of an elliptic PDE in n variables $\mathbf{x} = (x_1, \dots, x_n)^\top$ is
$$\Delta u + \sum_{i=1}^{n} b_i u_{x_i} + fu = g.$$
2. The normal form of a hyperbolic PDE in $n+1$ variables $(\mathbf{x}, t) = (x_1, \dots, x_n, t)^\top$ is
$$u_{tt} - \Delta u + \sum_{i=1}^{n} b_i u_{x_i} + fu = g.$$
3. The normal form of a parabolic PDE in $n+1$ variables $(\mathbf{x}, t) = (x_1, \dots, x_n, t)^\top$ is
$$\Delta u + b_0 u_t + \sum_{i=1}^{n} b_i u_{x_i} + fu = g.$$

Remark: in all cases Δ denotes the Laplace operator w.r.t. $\mathbf{x}$.

Examples:

1. The elliptic Laplace equation
$$\Delta u = 0.$$
2. The hyperbolic wave equation
$$u_{tt} - \Delta u = 0.$$
3. The parabolic heat equation
$$u_t = \Delta u.$$

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1. The normal form of an elliptic PDE in n variables $\mathbf{x} = (x_1, \ldots, x_n)^\top$ is

$$\Delta u + \sum_{i=1}^{n} b_i u_{x_i} + fu = g.$$

2. The normal form of a hyperbolic PDE in $n+1$ variables $(\mathbf{x}, t) = (x_1, \ldots, x_n, t)^\top$ is

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3. The normal form of a parabolic PDE in $n+1$ variables $(\mathbf{x}, t) = (x_1, \ldots, x_n, t)^\top$ is

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1. The elliptic Laplace equation

$$\Delta u = 0.$$

2. The hyperbolic wave equation

$$u_{tt} - \Delta u = 0.$$

3. The parabolic heat equation

$$u_t = \Delta u.$$

Well-Posedness

Definition: (Well-Posed Problem)
A correctly posed problem (or well-posed problem) consists of

- a *partial differential equation*, defined on a domain, and
- a set of *initial and/or boundary conditions*,

such that the following properties are fulfilled:

1. Existence: There exists at least one solution, that fulfills all above conditions;
2. Uniqueness: The solution is unique;
3. Stability: The solution depends cont. on the initial/boundary conditions
   i.e. small perturbations in the data yield small perturbations in the solution.

3

Example: (Well-Posed Wave Equation)
The initial value problem for the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{in } [0,\infty[\times \mathbb{R} \qquad (1)$$
$$u = f \quad \text{on } \{t=0\} \times \mathbb{R} \qquad (2)$$
$$u_t = g \quad \text{on } \{t=0\} \times \mathbb{R} \qquad (3)$$

is a well-posed hyperbolic problem.

- The uniquely determined solution is given by the representation by d'Alembert:

$$u(t,x) = \frac{1}{2}\left(f(x-ct) + f(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(z)dz\right).$$

- The solution depends continuously on the data, since

$$\|u - \tilde{u}\|_\infty \leq \|f - \tilde{f}\|_\infty + t\|g - \tilde{g}\|_\infty.$$

Example: (Hadamard)
The initial value problem for the PDE

$$u_{xx} + u_{yy} = 0 \quad \text{in } \mathbb{R}^2 \qquad (1)$$
$$u = f \quad \text{on } \mathbb{R} \times \{y=0\} \qquad (2)$$
$$u_y = g \quad \text{on } \mathbb{R} \times \{y=0\} \qquad (3)$$

is not a well-posed elliptic problem!

Example: (Laplace Equation)
The boundary value problem for the two-dimensional Laplace equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{in } \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \\ u &= g \quad \text{on } \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \end{aligned}$$

is a well-posed elliptic problem.
The unique solution is given by Poisson's integral form:

$$u(x,y) = \frac{1 - x^2 - y^2}{2\pi} \oint_{\|\mathbf{z}\|=1} \frac{g(\mathbf{z})}{\|\mathbf{x} - \mathbf{z}\|^2} ds$$

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is not a well-posed elliptic problem!

Reasoning:

- Set $f(x) = g(x) = 0$, then the unique solution is given by

$$u(x, y) = 0.$$

- On the other hand, if $f_n(x) = 0$ and $g_n(x) = \frac{1}{n}\sin(nx)$, for $n \in \mathbb{N}$, the solution is

$$u_n(x, y) = \frac{1}{n^2}\sin(nx)\sinh(ny)$$

- We have

$$\lim_{n\to\infty} f_n = f = 0 \quad \text{and} \quad \lim_{n\to\infty} g_n = g = 0$$

- But, since $\lim_{n\to\infty} \frac{1}{n^2}\sinh(ny) = \infty$ $(y \neq 0)$, we have

$$\lim_{n\to\infty} u_n \neq u,$$

thus the solution does not depend continuously on the data!

Example: (Laplace Equation)
The boundary value problem for the two-dimensional Laplace equation

$$\begin{array}{rcll} u_{xx} + u_{yy} & = & 0 & \text{in } \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \\ u & = & g & \text{on } \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \end{array}$$

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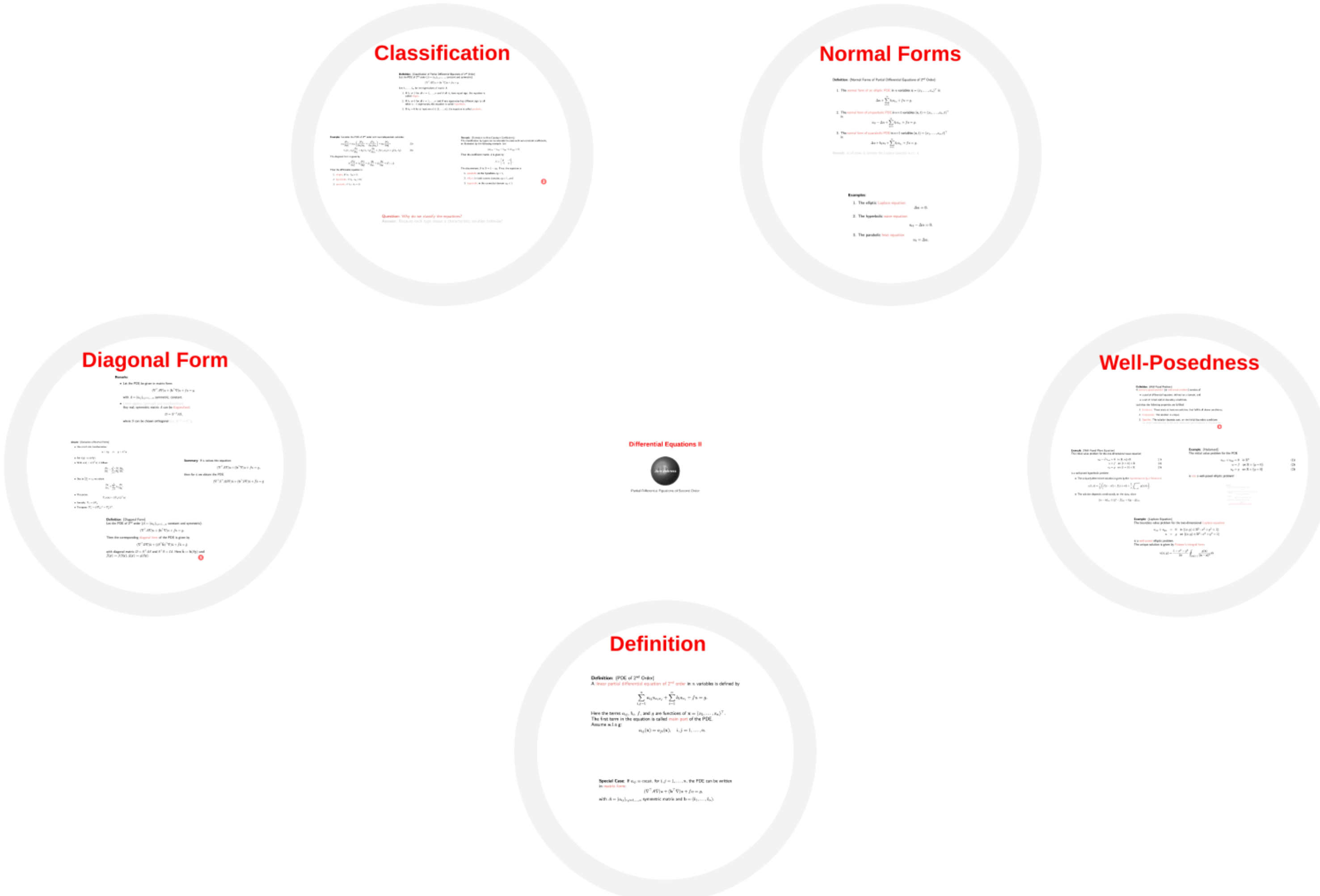
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