

# Differential Equations II



Laplace's Equation

# Recap

**Definition: (PDE of 2<sup>nd</sup> Order)**  
 A linear partial differential equation of 2<sup>nd</sup> order in  $n$  variables is defined by

$$\sum_{i,j=1}^n a_{ij}u_{x_i x_j} + \sum_{i=1}^n b_i u_{x_i} + f u = g.$$

Here the terms  $a_{ij}$ ,  $b_i$ ,  $f$ , and  $g$  are functions of  $\mathbf{x} = (x_1, \dots, x_n)^T$ .  
 The first term in the equation is called **main part** of the PDE.  
 Assume w.l.o.g.:

$$a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}), \quad i, j = 1, \dots, n.$$

**Definition: (Diagonal Form)**  
 Let the PDE of 2<sup>nd</sup> order ( $A = (a_{ij})_{i,j=1,\dots,n}$  constant and symmetric)

$$(\nabla^T A \nabla)u + (\mathbf{b}^T \nabla)u + f u = g.$$

Then the corresponding **diagonal form** of the PDE is given by

$$(\nabla^T D \nabla)\tilde{u} + ((S^T \mathbf{b})^T \nabla)\tilde{u} + \tilde{f}\tilde{u} = \tilde{g}$$

with diagonal matrix  $D = S^T A S$  and  $S^T S = Id$ . Here  $\tilde{\mathbf{b}} := \mathbf{b}(S\mathbf{y})$  and  $\tilde{f}(\mathbf{y}) := f(S\mathbf{y})$ ,  $\tilde{g}(\mathbf{y}) := g(S\mathbf{y})$ .

**Definition: (Classification of Partial Differential Equations of 2<sup>nd</sup> Order)**  
 Let the PDE of 2<sup>nd</sup> order ( $A = (a_{ij})_{i,j=1,\dots,n}$  constant and symmetric)

$$(\nabla^T A \nabla)u + (\mathbf{b}^T \nabla)u + f u = g.$$

Let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues of matrix  $A$ .

1. If  $\lambda_i \neq 0$  for all  $i = 1, \dots, n$  and if all  $\lambda_i$  have equal sign, the equation is called **elliptic**.
2. If  $\lambda_i \neq 0$  for all  $i = 1, \dots, n$  and if one eigenvalue has different sign to all other  $n - 1$  eigenvalues, the equation is called **hyperbolic**.
3. If  $\lambda_k = 0$  for at least one  $k \in \{1, \dots, n\}$ , the equation is called **parabolic**.

**Definition: (Well-Posed Problem)**  
 A **correctly posed problem** (or **well-posed problem**) consists of

- a **partial differential equation**, defined on a domain, and
- a set of **initial and/or boundary conditions**.

such that the following properties are fulfilled:

1. **Existence:** There exists at least one solution, that fulfills all above conditions;
2. **Uniqueness:** The solution is unique;
3. **Stability:** The solution depends **cont. on the initial/boundary conditions**  
 i.e. small perturbations in the data yield small perturbations in the solution.

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$$a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}), \quad i, j = 1, \dots, n.$$

**Definition:** (Diagonal Form)

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i.e. small perturbations in the data yield small perturbations in the solution.

# Introduction

**Definition: (Laplace's and Poisson's Equation)**  
Let  $u \in C^2(\mathbb{R}^n)$  be a twice cont. differentiable function,  $x \in D \subset \mathbb{R}^n$  open,  
 $u = u(x)$ . Then Laplace's equation is given by

$$\Delta u = 0.$$

Poisson's equation is defined as  
 $-\Delta u = f$   
with a given right hand side  $f = f(x)$ .

**Proposition: (Representation of Solution of Poisson's Equation)**  
A solution to Poisson's equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^n$$

is given by

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y) dy.$$

**Definition: (Harmonic Function)**  
Let  $u \in C^2(\mathbb{R}^n)$  be a twice cont. differentiable function satisfying Laplace's equation

$$\Delta u = 0.$$

Then  $u$  is called harmonic function.

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Example:

For  $x \in \mathbb{R}^3$  it holds:

$$\text{vol}(K_1(0)) = \alpha(3) = \frac{4\pi}{3}.$$

Thus, the fundamental solution of Laplace's equation is

$$\Phi(x) = \frac{1}{4\pi} \frac{1}{|x|}.$$

**Definition: (Fundamental Solution of Laplace's Equation)**  
The function  $\Phi(x)$ ,  $x \in \mathbb{R}^n$ ,  $x \neq 0$ , given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log \|x\| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \|x\|^{2-n} & (n \geq 3) \end{cases}$$

is called fundamental solution of Laplace's equation.

**Definition:** (Laplace's and Poisson's Equation)

Let  $u \in C^2(\mathbb{R}^n)$  be a twice cont. differentiable function,  $\mathbf{x} \in D \subset \mathbb{R}^n$  open,  $u = u(\mathbf{x})$ . Then **Laplace's equation** is given by

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**Definition:** (Harmonic Function)

Let  $u \in C^2(\mathbb{R}^n)$  be a twice cont. differentiable function satisfying Laplace's equation

$$\Delta u = 0.$$

Then  $u$  is called **harmonic function**.



**Definition:** (Fundamental Solution of Laplace's Equation)

The function  $\Phi(\mathbf{x})$ ,  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{x} \neq 0$ , given by

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\pi} \log \|\mathbf{x}\| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \|\mathbf{x}\|^{2-n} & (n \geq 3) \end{cases}$$

is called **fundamental solution of Laplace's equation**.

Remarks:

- The constant  $\alpha(n)$  denotes the volume of the unit ball in  $\mathbb{R}^n$ .
- The fundamental solution is a harmonic function for all  $\mathbf{x} \neq 0$ .

**Example:**

For  $\mathbf{x} \in \mathbb{R}^3$  it holds:

$$\text{vol}(K_1(0)) = \alpha(3) = \frac{4\pi}{3}.$$

Thus, the fundamental solution of Laplace's equation is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \frac{1}{\|\mathbf{x}\|}.$$

**Proposition:** (Representation of Solution of Poisson's Equation)

A solution to Poisson's equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^n$$

is given by

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) \, d\mathbf{y}.$$

# Properties of Harmonic Functions

**Proposition:** (Mean Value Property of Harmonic Functions)

Let  $U \subset \mathbb{R}^n$  be an open set. If  $u \in C^2(U)$  is harmonic, then for each ball  $B(x, r) \subset U$

$$u(x) = \int_{\partial B(x,r)} u \, dS = \int_{B(x,r)} \Delta u \, dy.$$

Here  $f$  denotes the mean over the sphere or the ball, resp.:

$$f \dots = \frac{1}{\text{vol}(B(x, r))} \int \dots$$

The reverse of the proposition holds as well:

**Proposition:** For the function  $u \in C^2(U)$  let

$$u(x) = \int_{\partial B(x,r)} u \, dS$$

for each ball  $B(x, r) \subset U$ . Then  $u$  is harmonic.

**Proposition:** (Maximum Principle of Harmonic Functions)

Let  $u \in C^2(U) \cap C(\bar{U})$  harmonic in  $U$ . Then

1. The **maximum principle** holds:

$$\max_{\bar{U}} u(x) = \max_{\partial U} u(x).$$

2. The **strong maximum principle** holds: if  $U$  is connected and if a point  $x_0 \in U$  exists with

$$u(x_0) = \max_{\bar{U}} u(x),$$

then  $u$  is constant on  $U$ .

**Propositions:**

- If a continuous function  $u \in C(\bar{U})$  on an open set  $U \subset \mathbb{R}^n$  satisfies the mean value property for each ball  $B(x, r) \subset U$ , then  $u$  is indefinitely often differentiable ( $u \in C^\infty(U)$ ).
- **Proposition of Liouville:** Let the function  $u: \mathbb{R}^n \rightarrow \mathbb{R}$  be harmonic and bounded. Then it holds that  $u$  is constant on the entire  $\mathbb{R}^n$ .
- **Bounded Solution of Poisson's Equation:** Let  $f \in C_0^2(\mathbb{R}^n)$ ,  $n \geq 3$ . Then each bounded solution of Poisson's equation  $-\Delta u = f$  in  $\mathbb{R}^n$  has the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y) \, dy + C$$

with a constant  $C$ .

**Proposition:** (Unique Solvability of Boundary Value Problem)

Let  $g \in C(\partial U)$  and  $f \in C(U)$ . Then there is at most one solution  $u \in C^2(U) \cap C(\bar{U})$  of the boundary value problem

$$\begin{aligned} -\Delta u &= f \text{ in } U \\ u &= g \text{ on } \partial U. \end{aligned}$$

**Proposition:** (Mean Value Property of Harmonic Functions)

Let  $U \subset \mathbb{R}^n$  be an open set. If  $u \in C^2(U)$  is harmonic, then for each ball  $B(\mathbf{x}, r) \subset U$

$$u(\mathbf{x}) = \int_{\partial B(\mathbf{x}, r)} u \, dS = \int_{B(\mathbf{x}, r)} u \, d\mathbf{y}.$$

Here  $\int$  denotes the mean over the sphere or the ball, resp.:

$$\int \dots = \frac{1}{\text{vol}(B(\mathbf{x}, r))} \int \dots$$

**Interpretation:** The mean value property means that the value of a harmonic function at position  $\mathbf{x}$  is equal to

- the mean of the function over a ball with center point  $\mathbf{x}$ , or
- the mean of the function over a corresponding sphere around  $\mathbf{x}$ .

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The reverse of the proposition holds as well:

**Proposition:** For the function  $u \in C^2(U)$  let

$$u(\mathbf{x}) = \int_{\partial B(\mathbf{x}, r)} u \, dS$$

for each ball  $B(\mathbf{x}, r) \subset U$ . Then  $u$  is harmonic.

**Proof:** (by contradiction)

- Let  $\Delta u \neq 0$ .
- Then a ball  $B(\mathbf{x}, r) \subset U$  exists, such that  $\Delta u > 0$  inside of  $B(\mathbf{x}, r)$ .
- On the other hand it holds  $0 = \phi'(r) = \frac{r}{n} \int_{B(\mathbf{x}, r)} \Delta u(\mathbf{y}) \, d\mathbf{y} > 0$ .
- This is a contradiction, i.e.  $u$  is harmonic.



**Proposition:** (Maximum Principle of Harmonic Functions)

Let  $u \in C^2(U) \cap C(\bar{U})$  harmonic in  $U$ . Then

1. The **maximum principle** holds:

$$\max_{x \in \bar{U}} u(\mathbf{x}) = \max_{x \in \partial U} u(\mathbf{x}).$$

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then  $u$  is constant on  $U$ .

**Idea of proof:** Appropriate application of mean value property...

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$$\begin{aligned} -\Delta u &= f && \text{in } U \\ u &= g && \text{on } \partial U. \end{aligned}$$

**Proof:**

- **Assumption:** Let  $u_1$  and  $u_2$  be two solutions.
- **Then:**  $w := \pm(u_1 - u_2)$  solves the boundary value problem

$$\begin{aligned} -\Delta w &= 0 && \text{in } U \\ w &= 0 && \text{on } \partial U. \end{aligned}$$

- **Maximum principle:** It holds:  $\pm(u_1 - u_2) = w \equiv 0$  on  $U$ .
- **Thus:**  $u_1 = u_2$ .

## Properties:

- If a continuous function  $u \in C(U)$  on an open set  $U \subset \mathbb{R}^n$  satisfies the *mean value property* for each ball  $B(\mathbf{x}, r) \subset U$ , then  $u$  is indefinitely often differentiable ( $u \in C^\infty(U)$ ).
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with a constant  $C$ .

# Boundary Value Problems

**Definitions:** (Dirichlet and Neumann Problem)

- The boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U \end{aligned}$$

is called **Dirichlet problem** of Poisson's equation  
(resp. of Laplace's equation, if  $f = 0$ ).

- The boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ \frac{\partial u}{\partial \mathbf{n}} &= g \quad \text{on } \partial U \end{aligned}$$

is called **Neumann problem** of Poisson's (resp. Laplace's) equation.  
Here  $\mathbf{n}$  is the outer normal at  $\partial U$ .

**Proposition:** Let  $u \in C^2(\bar{U})$ ,  $U \subset \mathbb{R}^n$  open set. Then for all  $\mathbf{x} \in U$  the relation holds

$$\begin{aligned} u(\mathbf{x}) &= \int_{\partial U} (\Phi(\mathbf{y} - \mathbf{x}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) - u(\mathbf{y}) \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{y} - \mathbf{x})) dS(\mathbf{y}) \\ &\quad - \int_U \Phi(\mathbf{y} - \mathbf{x}) \Delta u(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

The function  $\Phi$  denotes the fundamental solution of Laplace's equation.

## Definitions: (Dirichlet and Neumann Problem)

- The boundary value problem

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The function  $\Phi$  denotes the fundamental solution of Laplace's equation.

**Proof:** Use Green's formulas from Analysis III.

**Application:** For solving Laplace's and Poisson's equation:

In principle we are able to solve in each point,  
but we need boundary values for  $u$  and also for  $\frac{\partial u}{\partial \mathbf{n}}$ .

# Green's Functions

**Definition:** (Green's Function)

Let  $U \subset \mathbb{R}^n$  be open and  $\Phi^x(y)$  the solution of Dirichlet's Problem

$$\begin{aligned}\Delta \Phi^x &= 0 \quad \text{in } U \\ \Phi^x &= \Phi(y-x) \quad \text{on } \partial U.\end{aligned}$$

Then **Green's function**  $G$  on  $U$  is defined by

$$G(x, y) := \Phi(y-x) - \Phi^x(y) \quad x, y \in U, x \neq y.$$

**Proposition:** (Solution of Dirichlet Problem of Poisson's Equation)

Let  $u \in C^2(\bar{U})$  be a solution of the Dirichlet problem of Poisson's equation. Then  $u$  can be represented as

$$u(x) = \int_{\partial U} g(y) \frac{\partial G}{\partial \mathbf{n}}(x, y) \, dS(y) + \int_U f(y) G(x, y) \, dV(y) \quad (x \in U).$$

$f$  and  $g$  are the right hand side and boundary condition of the Dirichlet problem.

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**Remarks:** (Properties of Green's Function  $G(x, y)$ )

1.  $G(x, y)$  is harmonic in  $y$ , except for the point  $y = x$
2.  $G(x, y)$  satisfies homogeneous boundary conditions:

$$G(x, y) = 0 \quad \forall y \in \partial U, x \in U$$

3.  $G(x, y)$  is uniquely defined
4.  $G(x, y)$  is symmetric:

$$G(x, y) = G(y, x)$$

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4.  $G(\mathbf{x}, \mathbf{y})$  is symmetric:

$$G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x})$$

## Properties of Harmonic Functions

**Proposition:** (Mean Value Property of Harmonic Functions)  
Let  $f$  be a harmonic function on a domain  $D \subset \mathbb{R}^n$ . If  $B_r(x_0) \subset D$ , then  
$$f(x_0) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x_0)} f(x) dS(x)$$
  
and  
$$f(x_0) = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} \Delta f(x) dx$$

**Proposition:** (Maximum Principle for Harmonic Functions)  
Let  $f$  be a harmonic function on a domain  $D \subset \mathbb{R}^n$ . If  $D$  is bounded, then the maximum and minimum of  $f$  are attained on the boundary  $\partial D$ .

**Proposition:** (Liouville's Theorem)  
If  $f$  is a bounded harmonic function on  $\mathbb{R}^n$ , then  $f$  is constant.

## Introduction

**Definition:** (Laplace's Equation)  
A function  $u$  is harmonic in a domain  $D \subset \mathbb{R}^n$  if it satisfies Laplace's equation  $\Delta u = 0$  in  $D$ .

**Proposition:** (Representation of Harmonic Functions)  
Let  $u$  be a harmonic function in a domain  $D \subset \mathbb{R}^n$ . Then  $u$  can be represented as the real part of a holomorphic function in  $D$ .

**Definition:** (Dirichlet Problem)  
The Dirichlet problem for a domain  $D \subset \mathbb{R}^n$  is to find a harmonic function  $u$  in  $D$  such that  $u = g$  on  $\partial D$ .

**Proposition:** (Uniqueness of Solution to the Dirichlet Problem)  
If  $g$  is a continuous function on  $\partial D$ , then there is at most one harmonic function  $u$  in  $D$  such that  $u = g$  on  $\partial D$ .

## Differential Equations II



## Recap

**Definition:** (Eigenvalue Problem)  
A function  $u$  is an eigenfunction of the Laplace operator  $\Delta$  with eigenvalue  $\lambda$  if  $\Delta u = \lambda u$ .

**Proposition:** (Sturm-Liouville Theory)  
Let  $L$  be a self-adjoint operator on a Hilbert space  $H$ . Then the eigenvalues of  $L$  are real and the corresponding eigenfunctions are orthogonal.

**Definition:** (Green's Function)  
The Green's function  $G(x, y)$  for a domain  $D \subset \mathbb{R}^n$  is a function that satisfies  $\Delta G = \delta(x - y)$  in  $D$  and  $G = 0$  on  $\partial D$ .

## Boundary Value Problems

**Definition:** (Dirichlet and Neumann Problems)  
The Dirichlet problem for a domain  $D \subset \mathbb{R}^n$  is to find a harmonic function  $u$  in  $D$  such that  $u = g$  on  $\partial D$ . The Neumann problem is to find a harmonic function  $u$  in  $D$  such that  $\frac{\partial u}{\partial n} = g$  on  $\partial D$ .

**Proposition:** (Existence and Uniqueness of Solutions)  
The Dirichlet problem has a unique solution if  $g$  is continuous on  $\partial D$ . The Neumann problem has a unique solution if  $g$  is continuous on  $\partial D$  and  $\int_{\partial D} g = 0$ .

## Green's Functions

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**Proposition:** (Representation of Harmonic Functions)  
Let  $u$  be a harmonic function in a domain  $D \subset \mathbb{R}^n$ . Then  $u$  can be represented as  $u(x) = \int_{\partial D} u(y) \frac{\partial G(x, y)}{\partial n} dy$ .

**Definition:** (Poisson's Kernel)  
The Poisson's kernel  $P(x, y)$  for a domain  $D \subset \mathbb{R}^n$  is a function that satisfies  $\Delta P = 0$  in  $D$  and  $P = 0$  on  $\partial D$ .