Differential Equations II

Laplace's Equation

Recap

Definition: (PDE of 2nd Order)
A linear partial differential equation of 2nd order in n variables is defined by

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + fu = g.$$

Here the terms a_{ij}, b_i, f, and g are functions of $\mathbf{x} = (x_1, \dots, x_n)^\top$.
The first term in the equation is called main part of the PDE.
Assume w.l.o.g:

$$a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x}), \quad i, j = 1, \dots, n.$$

Definition: (Diagonal Form)
Let the PDE of 2nd order ($A = (a_{ij})_{i,j=1\dots,n}$ constant and symmetric)

$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g.$$

Then the corresponding diagonal form of the PDE is given by

$$(\nabla^\top D \nabla)\tilde{u} + ((S^\top \tilde{\mathbf{b}})^\top \nabla)\tilde{u} + \tilde{f}\tilde{u} = \tilde{g}$$

with diagonal matrix $D = S^\top A S$ and $S^\top S = Id$. Here $\tilde{\mathbf{b}} := \mathbf{b}(S\mathbf{y})$ and $\tilde{f}(\mathbf{y}) := f(S\mathbf{y})$, $\tilde{g}(\mathbf{y}) := g(S\mathbf{y})$.

Definition: (Classification of Partial Differential Equations of 2nd Order)
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$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g.$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of matrix A.

1. If $\lambda_i \neq 0$ for all $i = 1, \dots, n$ and if all λ_i have equal sign, the equation is called elliptic.
2. If $\lambda_i \neq 0$ for all $i = 1, \dots, n$ and if one eigenvalue has different sign to all other $n-1$ eigenvalues, the equation is called hyperbolic.
3. If $\lambda_k = 0$ for at least one $k \in \{1, \dots, n\}$, the equation is called parabolic.

Definition: (Well-Posed Problem)
A correctly posed problem (or well-posed problem) consists of

- a *partial differential equation*, defined on a domain, and
- a set of *initial and/or boundary conditions*,

such that the following properties are fulfilled:

1. Existence: There exists at least one solution, that fulfills all above conditions;
2. Uniqueness: The solution is unique;
3. Stability: The solution depends cont. on the initial/boundary conditions
   i.e. small perturbations in the data yield small perturbations in the solution.

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Introduction

Definition: (Laplace's and Poisson's Equation)
Let $u \in C^2(\mathbb{R}^n)$ be a twice cont. differentiable function, $\mathbf{x} \in D \subset \mathbb{R}^n$ open, $u = u(\mathbf{x})$. Then Laplace's equation is given by

$$\Delta u = 0.$$

Poisson's equation is defined as

$$-\Delta u = f$$

with a given right hand side $f = f(\mathbf{x})$.

Proposition: (Representation of Solution of Poisson's Equation)
A solution to Poisson's equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^n$$

is given by

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}.$$

Definition: (Harmonic Function)
Let $u \in C^2(\mathbb{R}^n)$ be a twice cont. differentiable function satisfying Laplace's equation

$$\Delta u = 0.$$

Then u is called harmonic function.

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Example:
For $\mathbf{x} \in \mathbb{R}^3$ it holds:

$$\text{vol}(K_1(0)) = \alpha(3) = \frac{4\pi}{3}.$$

Thus, the fundamental solution of Laplace's equation is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \frac{1}{\|x\|}.$$

Definition: (Fundamental Solution of Laplace's Equation)
The function $\Phi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$, given by

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\pi} \log \|x\| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \|x\|^{2-n} & (n \geq 3) \end{cases}$$

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is called fundamental solution of Laplace's equation.

Remarks:

- The constant $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$.
- The fundamental solution is a harmonic function for all $\mathbf{x} \neq 0$.

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A solution to Poisson's equation

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$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}.$$

Properties of Harmonic Functions

Proposition: (Mean Value Property of Harmonic Functions)
Let $U \subset \mathbb{R}^n$ be an open set. If $u \in C^2(U)$ is harmonic, then for each ball $B(\mathbf{x}, r) \subset U$

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} u \, dS = \fint_{B(\mathbf{x},r)} u \, d\mathbf{y}.$$

Here $\fint$ denotes the mean over the shpere or the ball, resp.:

$$\fint \cdots = \frac{1}{\mathrm{vol}(B(\mathbf{x}, r))} \int \cdots$$

The reverse of the proposition holds as well:
Proposition: For the function $u \in C^2(U)$ let

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} u \, dS$$

for each ball $B(\mathbf{x}, r) \subset U$. Then u is harmonic.

Propoerties:

- If a continuous function $u \in C(U)$ on an open set $U \subset \mathbb{R}^n$ satisfies the *mean value property* for each ball $B(\mathbf{x}, r) \subset U$, then u is indefinitely often differentiable ($u \in C^\infty(U)$).
- Proposition of Liouville: Let the function $u : \mathbb{R}^n \to \mathbb{R}$ be harmonic and bounded. Then it holds that u is constant on the entire $\mathbb{R}^n$.
- Bounded Solution of Poisson's Equation: Let $f \in C_c^2(\mathbb{R}^n)$, $n \geq 3$. Then each bounded solution of Poisson's equation $-\Delta u = f$ in $\mathbb{R}^n$ has the form

  $$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) \, d\mathbf{y} + C$$

  with a constant C.

Proposition: (Maximum Principle of Harmonic Functions)
Let $u \in C^2(U) \cap C(\overline{U})$ harmonic in U. Then

1. The maximum principle holds:

   $$\max_{x \in \overline{U}} u(\mathbf{x}) = \max_{x \in \partial U} u(\mathbf{x}).$$

2. The strong maximum principle holds: is U connected and if a point $x_0 \in U$ exists with

   $$u(x_0) = \max_{x \in \overline{U}} u(\mathbf{x}),$$

   then u is constant on U.

Proposition: (Unique Solvability of Boundary Value Problem)
Let $g \in C(\partial U)$ and $f \in C(U)$. Then there is at most one solution $u \in C^2(U) \cap C(\overline{U})$ of the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U. \end{aligned}$$

Proposition: (Mean Value Property of Harmonic Functions)
Let $U \subset \mathbb{R}^n$ be an open set. If $u \in C^2(U)$ is harmonic, then for each ball $B(\mathbf{x}, r) \subset U$

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Here $\fint$ denotes the mean over the shpere or the ball, resp.:

$$\fint \cdots = \frac{1}{\mathrm{vol}(B(\mathbf{x}, r))} \int \cdots$$

Interpretation: The mean value property means that the value of a harmonic function at position $\mathbf{x}$ is equal to

- the mean of the function over a ball with center point $\mathbf{x}$, or
- the mean of the function over a corresponding sphere around $\mathbf{x}$.

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The reverse of the proposition holds as well:

Proposition: For the function $u \in C^2(U)$ let

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for each ball $B(\mathbf{x}, r) \subset U$. Then u is harmonic.

Proof: (by contradiction)

- Let $\Delta u \neq 0$.
- Then a ball $B(\mathbf{x}, r) \subset U$ exists, such that $\Delta u > 0$ inside of $B(\mathbf{x}, r)$.
- On the other hand it holds $0 = \phi'(r) = \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u(\mathbf{y})\, d\mathbf{y} > 0$.
- This is a contradiction, i.e. u is harmonic.

Proposition: (Maximum Principle of Harmonic Functions)
Let $u \in C^2(U) \cap C(\overline{U})$ harmonic in U. Then

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Idea of proof: Appropriate application of mean value property...

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Let $g \in C(\partial U)$ and $f \in C(U)$. Then there is at most one solution $u \in C^2(U) \cap C(\overline{U})$ of the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U. \end{aligned}$$

Proof:

- **Assumption**: Let u_1 and u_2 be two solutions.
- **Then**: $w := \pm(u_1 - u_2)$ solves the boundary value problem

$$\begin{aligned} -\Delta w &= 0 \quad \text{in } U \\ w &= 0 \quad \text{on } \partial U. \end{aligned}$$

- **Maximum principle**: It holds: $\pm(u_1 - u_2) = w \equiv 0$ on U.
- **Thus**: $u_1 = u_2$.

Propoerties:

- If a continuous function $u \in C(U)$ on an open set $U \subset \mathbb{R}^n$ satisfies the *mean value property* for each ball $B(\mathbf{x}, r) \subset U$, then u is indefinitely often differentiable ($u \in C^\infty(U)$).

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  with a constant C.

Boundary Value Problems

Definitions: (Dirichlet and Neumann Problem)

- The boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U \end{aligned}$$

  is called Dirichlet problem of Poisson's equation (resp. of Laplace's equation, if $f = 0$).

- The boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ \frac{\partial u}{\partial \mathbf{n}} &= g \quad \text{on } \partial U \end{aligned}$$

  is called Neumann problem of Poisson's (resp. Laplace's) equation. Here $\mathbf{n}$ is the outer normal at ∂U.

Proposition: Let $u \in C^2(\overline{U})$, $U \subset \mathbb{R}^n$ open set. Then for all $\mathbf{x} \in U$ the relation holds

$$\begin{aligned} u(x) &= \int_{\partial U} (\Phi(\mathbf{y}-\mathbf{x}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) - u(\mathbf{y}) \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{y}-\mathbf{x}))\, dS(\mathbf{y}) \\ &- \int_U \Phi(\mathbf{y}-\mathbf{x}) \Delta u(\mathbf{y})\, d\mathbf{y}. \end{aligned}$$

The function Φ denotes the fundamental solution of Laplace's equation.

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&- \int_U \Phi(\mathbf{y}-\mathbf{x})\Delta u(\mathbf{y})\, d\mathbf{y}.
\end{aligned}
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The function Φ denotes the fundamental solution of Laplace's equation.

Proof: Use Green's formulas from Analysis III.

Application: For solving Laplace's and Poisson's equation:
In principle we are able to solve in each point,
but we need boundary values for u and also for $\frac{\partial u}{\partial \mathbf{n}}$.

Green's Functions

Definition: (Green's Function)
Let $U \subset \mathbb{R}^n$ be open and $\Phi^x(\mathbf{y})$ the solution of Dirichlet's Problem

$$\begin{aligned} \Delta\Phi^x &= 0 \quad \text{in } U \\ \Phi^x &= \Phi(\mathbf{y}-\mathbf{x}) \quad \text{on } \partial U. \end{aligned}$$

Then Green's function G on U is defined by

$$G(\mathbf{x},\mathbf{y}) := \Phi(\mathbf{y}-\mathbf{x}) - \Phi^x(\mathbf{y}) \quad \mathbf{x},\mathbf{y} \in U, \mathbf{x} \neq \mathbf{y}.$$

Proposition: (Solution of Dirichlet Problem of Poisson's Equation)
Let $u \in C^2(\overline{U})$ be a solution of the Dirichlet problem of Poisson's equation. Then u can be represented as

$$u(\mathbf{x}) = \int_{\partial U} g(\mathbf{y}) \frac{\partial G}{\partial \mathbf{n}}(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}) + \int_U f(\mathbf{y}) G(\mathbf{x},\mathbf{y})\, d\mathbf{y} \quad (\mathbf{x} \in U).$$

f and g are the right hand side and boundary condition of the Dirichlet problem.

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Remarks: (Properties of Green's Function $G(\mathbf{x},\mathbf{y})$)

1. $G(\mathbf{x},\mathbf{y})$ is harmonic in $\mathbf{y}$, except for the point $\mathbf{y} = \mathbf{x}$
2. $G(\mathbf{x},\mathbf{y})$ satisfies homogeneous boundary conditions:
$$G(\mathbf{x},\mathbf{y}) = 0 \quad \forall \mathbf{y} \in \partial U, \mathbf{x} \in U$$
3. $G(\mathbf{x},\mathbf{y})$ is uniquely defined
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