

Differential Equations II



Green's Function

Recap

Definition: (Green's Function)

Let $U \subset \mathbb{R}^n$ be open and $\Phi^x(\mathbf{y})$ the solution of Dirichlet's Problem

$$\begin{aligned}\Delta \Phi^x &= 0 \quad \text{in } U \\ \Phi^x &= \Phi(\mathbf{y} - \mathbf{x}) \quad \text{on } \partial U.\end{aligned}$$

Then **Green's function** G on U is defined by

$$G(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{y} - \mathbf{x}) - \Phi^x(\mathbf{y}) \quad \mathbf{x}, \mathbf{y} \in U, \mathbf{x} \neq \mathbf{y}.$$

Proposition: (Solution of Dirichlet Problem of Poisson's Equation)

Let $u \in C^2(\bar{U})$ be a solution of the Dirichlet problem of Poisson's equation. Then u can be represented as

$$u(\mathbf{x}) = \int_{\partial U} g(\mathbf{y}) \frac{\partial G}{\partial \mathbf{n}}(\mathbf{x}, \mathbf{y}) \, dS(\mathbf{y}) + \int_U f(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \quad (\mathbf{x} \in U).$$

f and g are the right hand side and boundary condition of the Dirichlet problem.

Remarks: (Properties of Green's Function $G(\mathbf{x}, \mathbf{y})$)

1. $G(\mathbf{x}, \mathbf{y})$ is harmonic in \mathbf{y} , except for the point $\mathbf{y} = \mathbf{x}$
2. $G(\mathbf{x}, \mathbf{y})$ satisfies homogeneous boundary conditions:

$$G(\mathbf{x}, \mathbf{y}) = 0 \quad \forall \mathbf{y} \in \partial U, \mathbf{x} \in U$$

3. $G(\mathbf{x}, \mathbf{y})$ is uniquely defined

4. $G(\mathbf{x}, \mathbf{y})$ is symmetric:

$$G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x}) \quad \text{①}$$

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Green's Function and Poisson Kernel for Half Space \mathbb{R}_+^n

Definition: (Poisson Kernel)

The function

$$K(\mathbf{x}, \mathbf{y}) := \frac{2x_n}{n\alpha(n)} \frac{1}{|\mathbf{x} - \mathbf{y}|^n},$$

where $\mathbf{x} \in \mathbb{R}_+^n$, $\mathbf{y} \in \partial\mathbb{R}_+^n$ is called **Poisson Kernel** of \mathbb{R}_+^n .

Proposition: (Dirichlet Problem for Laplace's Equation)

Let the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^n \\ u = g & \text{on } \partial\mathbb{R}_+^n = \{\mathbf{x} = (x_1, \dots, x_n)^\top : x_n = 0\} \end{cases}$$

be given. Then the solution is given by **Poisson's integral form**

$$u(\mathbf{x}) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}_+^n} \frac{g(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} d\mathbf{y}.$$

In particular, due to

$$\int_{\partial\mathbb{R}_+^n} K(\mathbf{x}, \mathbf{y}) d\mathbf{y} = 1$$

$u(\mathbf{x})$ is **bounded**, if g is bounded. Furthermore, one can show that u is **infinitely differentiable**.

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In particular, due to

$$\int_{\partial\mathbb{R}_+^n} K(\mathbf{x}, \mathbf{y}) d\mathbf{y} = 1$$

$u(\mathbf{x})$ is **bounded**, if g is bounded. Furthermore, one can show that u is **indefinitely differentiable**.

Green's Function and Poisson Kernel for Unit Ball

Proposition: (Dirichlet Problem for Laplace's Equation on the Unit Ball)
Let the boundary problem be given

$$\begin{cases} \Delta u = 0 & \text{in } \{x \in \mathbb{R}^n; |x| < 1\} \\ u = g & \text{on } \{x \in \mathbb{R}^n; |x| = 1\} \end{cases}$$

Then the solution is given by Poisson's Integral Form

$$u(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{|y|=1} \frac{g(y)}{|x - y|^n} dS(y).$$

Thus, the Poisson kernel for the unit ball is

$$K(x, y) = \frac{1 - |x|^2}{n\alpha(n)} \frac{1}{|x - y|^n}$$

for $|x| < 1$ and $|y| = 1$.

Reasoning

- Let $x \in \mathbb{R}^n \setminus \{0\}$. Then $k = \frac{x}{|x|}$ denotes the **dual point** of x w.r.t. the boundary of the unit ball $\partial B(0, 1)$.
- Therefore, the solution of the correction problem
$$\begin{cases} \Delta \Phi^* = 0 & \text{in } B(0, 1) = \{x \in \mathbb{R}^n; |x| < 1\} \\ \Phi^* = \Phi(y - x) & \text{on } \partial B(0, 1) \end{cases}$$
 is given by $\Phi^*(y) = \Phi(|x|(y - k))$.
- Obtain Green's function for the unit ball as $G(x, y) = \Phi(y - x) - \Phi(|x|(y - k))$ for $x, y \in B(0, 1)$, $x \neq y$.

Remark: Using the transformation

$$\tilde{u}(x) = u(rx)$$

a solution for the ball $B(0, r) = \{x; |x| < r\}$ can easily be derived.

Proposition: (Dirichlet Problem for Laplace's Equation on the Unit Ball)

Let the boundary problem be given

$$\begin{cases} \Delta u = 0 & \text{in } \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\} \\ u = g & \text{on } \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\} \end{cases}$$

Then the solution is given by **Poisson's Integral Form**

$$u(\mathbf{x}) = \frac{1 - |\mathbf{x}|^2}{n\alpha(n)} \int_{|\mathbf{y}|=1} \frac{g(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} dS(\mathbf{y}).$$

Thus, the Poisson kernel for the unit ball is

$$K(\mathbf{x}, \mathbf{y}) = \frac{1 - |\mathbf{x}|^2}{n\alpha(n)} \frac{1}{|\mathbf{x} - \mathbf{y}|^n}$$

for $|\mathbf{x}| < 1$ and $|\mathbf{y}| = 1$.

Reasoning:

- Let $\mathbf{x} \in \mathbb{R} \setminus \{0\}$. Then

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^2}$$

denotes the **dual point** of \mathbf{x} w.r.t. the boundary of the unit ball $\partial B(0, 1)$.

- Therefore, the solution of the correction problem

$$\begin{cases} \Delta \Phi^x = 0 & \text{in } \mathring{B}(0, 1) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\} \\ \Phi^x = \Phi(\mathbf{y} - \mathbf{x}) & \text{on } \partial B(0, 1) \end{cases}$$

is given by

$$\Phi^x(\mathbf{y}) = \Phi(|\mathbf{x}|(\mathbf{y} - \tilde{\mathbf{x}})).$$

- Obtain Green's function for the unit ball as

$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x}) - \Phi(|\mathbf{x}|(\mathbf{y} - \tilde{\mathbf{x}}))$$

for $\mathbf{x}, \mathbf{y} \in B(0, 1)$, $\mathbf{x} \neq \mathbf{y}$.

Remark: Using the transformation

$$\tilde{u}(\mathbf{x}) = u(r\mathbf{x})$$

a solution for the ball $B(0, r) = \{\mathbf{x} : |\mathbf{x}| < r\}$ can easily be derived.

Recap

Definition (Green's Function)
 Let $D \subset \mathbb{R}^n$ be open and $\partial D \neq \emptyset$ the solution of Dirichlet's Problem
 $\Delta u = 0$ in D
 $u = \phi$ on ∂D
 The Green's function $G(x, y)$ of D is defined by

$$G(x, y) = \phi(y) - \phi(x) - \phi(y), \quad x, y \in D, x \neq y.$$

Proposition (Representation of Green's Function of Poisson's Problem)
 Let $u \in C^2(D)$ be a solution of the Dirichlet problem of Poisson's equation. Then u can be represented as

$$u(x) = \int_D \frac{\Delta u(y)}{4\pi |x-y|} dy + \int_{\partial D} \phi(y) \frac{\partial G(x, y)}{\partial n(y)} dy$$

f and ϕ are the right-hand side and boundary condition of the Dirichlet problem.

Remarks (Properties of Green's Function $G(x, y)$)

- $G(x, y)$ is harmonic in y , except for the point $y = x$.
- $G(x, y)$ satisfies homogeneous boundary conditions: $G(x, y) = 0$ for $x \in \partial D$ or $y \in \partial D$.
- $G(x, y)$ is symmetric: $G(x, y) = G(y, x)$.
- $G(x, x)$ is well-defined: $G(x, x) = -\phi(x)$.

Green's Function and Poisson Kernel for Half Space \mathbb{R}_+^n

Definition (Poisson Kernel)
 The function

$$K(x, y) = \frac{2|x-y|}{|x-y|^2 + |x_1 - y_1|^2} \frac{1}{|x-y|^{n-2}}$$
 where $x \in \mathbb{R}_+^n$, $y \in \partial \mathbb{R}_+^n$ is called Poisson Kernel of \mathbb{R}_+^n .

Proposition (Dirichlet Problem for Laplace's Equation)
 Let the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^n \\ u = \phi & \text{on } \partial \mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_1 = 0, x_2, \dots, x_n \in \mathbb{R}\} \end{cases}$$
 be given. Then the solution is given by Poisson's integral formula

$$u(x) = \int_{\partial \mathbb{R}_+^n} \frac{K(x, y) \phi(y) dy}{\int_{\partial \mathbb{R}_+^n} K(x, y) dy}$$
 In particular, due to

$$\int_{\partial \mathbb{R}_+^n} K(x, y) dy = 1$$
 $K(x, \cdot)$ is harmonic, if y is bounded. Furthermore, we can show that it is infinitely differentiable.

Differential Equations II



Green's Function

Green's Function and Poisson Kernel for Unit Ball

Proposition (Dirichlet Problem for Laplace's Equation on the Unit Ball)
 Let $D = B_1(0) \subset \mathbb{R}^n$ be the unit ball in \mathbb{R}^n . Then the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1(0) \\ u = \phi & \text{on } \partial B_1(0) \end{cases}$$
 is given by Poisson's integral formula

$$u(x) = \int_{\partial B_1(0)} \frac{K(x, y) \phi(y) dy}{\int_{\partial B_1(0)} K(x, y) dy}$$
 where $K(x, y) = \frac{1 - |x|^2}{|x-y|^n}$ is the Poisson kernel for the unit ball.

Remark (Properties of Poisson Kernel $K(x, y)$)

- $K(x, y)$ is harmonic in y , except for the point $y = x$.
- $K(x, y)$ satisfies homogeneous boundary conditions: $K(x, y) = 0$ for $x \in \partial B_1(0)$ or $y \in \partial B_1(0)$.
- $K(x, y)$ is symmetric: $K(x, y) = K(y, x)$.
- $K(x, x)$ is well-defined: $K(x, x) = \frac{1 - |x|^2}{|x-x|^n}$.