Differential Equations II

Jörn Behrens

Green's Function

Recap

Definition: (Green's Function)
Let $U \subset \mathbb{R}^n$ be open and $\Phi^x(\mathbf{y})$ the solution of Dirichlet's Problem

$$\begin{aligned} \Delta\Phi^x &= 0 \quad \text{in } U \\ \Phi^x &= \Phi(\mathbf{y}-\mathbf{x}) \quad \text{on } \partial U. \end{aligned}$$

Then Green's function G on U is defined by

$$G(\mathbf{x},\mathbf{y}) := \Phi(\mathbf{y}-\mathbf{x}) - \Phi^x(\mathbf{y}) \quad \mathbf{x},\mathbf{y} \in U, \mathbf{x} \neq \mathbf{y}.$$

Proposition: (Solution of Dirichlet Problem of Poisson's Equation)
Let $u \in C^2(\overline{U})$ be a solution of the Dirichlet problem of Poisson's equation. Then u can be represented as

$$u(\mathbf{x}) = \int_{\partial U} g(\mathbf{y}) \frac{\partial G}{\partial \mathbf{n}}(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}) + \int_U f(\mathbf{y}) G(\mathbf{x},\mathbf{y})\, d\mathbf{y} \quad (\mathbf{x} \in U).$$

f and g are the right hand side and boundary condition of the Dirichlet problem.

Remarks: (Properties of Green's Function $G(\mathbf{x},\mathbf{y})$)

1. $G(\mathbf{x},\mathbf{y})$ is harmonic in $\mathbf{y}$, except for the point $\mathbf{y} = \mathbf{x}$
2. $G(\mathbf{x},\mathbf{y})$ satisfies homogeneous boundary conditions:
$$G(\mathbf{x},\mathbf{y}) = 0 \quad \forall \mathbf{y} \in \partial U, \mathbf{x} \in U$$
3. $G(\mathbf{x},\mathbf{y})$ is uniquely defined
4. $G(\mathbf{x},\mathbf{y})$ is symmetric:
$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x})$$ 1

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Green's Function and Poisson Kernel for Half Space $\mathbb{R}^n_+$

Definition: (Poisson Kernel)
The function

$$K(\mathbf{x},\mathbf{y}) := \frac{2x_n}{n\alpha(n)} \frac{1}{|\mathbf{x}-\mathbf{y}|^n},$$

where $\mathbf{x} \in \mathbb{R}^n_+$, $\mathbf{y} \in \partial\mathbb{R}^n_+$ is called Poisson Kernel of $\mathbb{R}^n_+$.

Proposition: (Dirichlet Problem for Laplace's Equation)
Let the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+ \\ u = g & \text{on} \quad \partial\mathbb{R}^n_+ = \{\mathbf{x} = (x_1, \ldots, x_n)^\top : x_n = 0\} \end{cases}$$

be given. Then the solution is given by Poisson's integral form

$$u(\mathbf{x}) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^n} \, d\mathbf{y}.$$

In particular, due to

$$\int_{\partial\mathbb{R}^n_+} K(\mathbf{x},\mathbf{y}) \, d\mathbf{y} = 1$$

$u(\mathbf{x})$ is bounded, if g is bounded. Furthermore, one can show that u is indefinitely differentiable.

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Green's Function and Poisson Kernel for Unit Ball

Proposition: (Dirichlet Problem for Laplace's Equation on the Unit Ball)
Let the boundary problem be given

$$\begin{cases} \Delta u = 0 & \text{in} \quad \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\} \\ u = g & \text{on} \quad \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\} \end{cases}$$

Then the solution is given by Poisson's Integral Form

$$u(\mathbf{x}) = \frac{1 - |\mathbf{x}|^2}{n\alpha(n)} \int_{|\mathbf{y}|=1} \frac{g(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} \, dS(\mathbf{y}).$$

Thus, the Poisson kernel for the unit ball is

$$K(\mathbf{x}, \mathbf{y}) = \frac{1 - |\mathbf{x}|^2}{n\alpha(n)} \frac{1}{|\mathbf{x} - \mathbf{y}|^n}$$

for $|\mathbf{x}| < 1$ and $|\mathbf{y}| = 1$.

Reasoning:

- Let $\mathbf{x} \in \mathbb{R} \setminus \{0\}$. Then
$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^2}$$
denotes the dual point of $\mathbf{x}$ w.r.t. the boundary of the unit ball $\partial B(0,1)$.
- Therefore, the solution of the correction problem
$$\begin{cases} \Delta \Phi^x = 0 & \text{in} \quad \mathring{B}(0,1) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\} \\ \Phi^x = \Phi(\mathbf{y} - \mathbf{x}) & \text{on} \quad \partial B(0,1) \end{cases}$$
is given by
$$\Phi^x(y) = \Phi(|\mathbf{x}|(\mathbf{y} - \tilde{\mathbf{x}})).$$
- Obtain Green's function for the unit ball as
$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x}) - \Phi(|\mathbf{x}|(\mathbf{y} - \tilde{\mathbf{x}}))$$
for $\mathbf{x}, \mathbf{y} \in B(0,1)$, $x \neq y$.

Remark: Using the transformation

$$\tilde{u}(\mathbf{x}) = u(r\mathbf{x})$$

a solution for the ball $B(0,r) = \{\mathbf{x} : |\mathbf{x}| < r\}$ can easily be derived.

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