Differential Equations II

Heat Equation

Heat Equation

Problem Formulation: (Heat Equation)
We look for explicit solution of the heat equation

$$u_t = \Delta_x u.$$

- $t \geq 0$ is a time variable;
- $\mathbf{x} \in U$, $U \subset \mathbb{R}^n$ open, is a spatial variable.

Initial Value Problem: (Cauchy Problem)
Let $U = \mathbb{R}^n$:

$$\begin{cases} u_t = & \Delta_x u \quad \text{in} \quad \mathbb{R}^n \times]0, T] \\ u = & g \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Summary: (Product Approach)

- Let the one-dimensional initial and boundary value problem be given

$$\begin{cases} u_t = & u_{xx} \quad \text{for} \quad 0 < t < \pi,\ 0 < t \leq T \\ u(x,0) = & \sin x \quad \text{for} \quad 0 \leq x \leq \pi \\ u(0,t) = u(\pi,t) = & 0 \quad \text{for} \quad 0 \leq t \leq T \end{cases}$$

- Use product ansatz for the solution:

$$u(x,t) = q(t) \cdot p(x).$$

- Obtain two ordinary differential equations:

$$\begin{aligned} \dot{q}(t) + \delta q(t) &= 0, \\ p''(x) + \delta p(x) &= 0. \end{aligned}$$

- Solution classes Depending on δ:

$$\begin{aligned} u(x,t) &= c_0 e^{-\delta t} \cdot (c_1 x + c_2) \\ u(x,t) &= c_0 e^{-\delta t} \cdot (c_1 e^{-\sqrt{|\delta|}x} + c_2 e^{\sqrt{|\delta|}x}) \\ u(x,t) &= c_0 e^{-\delta t} \cdot (c_1 \sin(\sqrt{\delta}x) + c_2 \cos(\sqrt{\delta}x)) \end{aligned}$$

- Parameters $\{c_0, c_1, c_2, \delta\}$ generally cannot be determined by initial and boundary values allone.

Problem Formulation: (Heat Equation)
We look for explicit solution of the heat equation

$$u_t = \Delta_x u.$$

- $t \geq 0$ is a time variable;
- $\mathbf{x} \in U$, $U \subset \mathbb{R}^n$ open, is a spatial variable.

Initial Value Problem: (Cauchy Problem)

Let $U = \mathbb{R}^n$:

$$\begin{cases} u_t = & \Delta_x u \quad \text{in} \quad \mathbb{R}^n \times]0,T] \\ u = & g \quad \text{on} \quad \mathbb{R}^n \times \{t=0\} \end{cases}$$

Initial and Boundary Value Problem:

Let $U \subset \mathbb{R}^n$ be bounded:

$$\begin{cases} u_t = & \Delta_x u \quad \text{in} \quad U_T := U \times]0,T] \\ u = & g \quad \text{on} \quad \Gamma_T := \overline{U_T} \backslash U_T \end{cases}$$

2

Summary: (Product Approach)

- Let the one-dimensional initial and boundary value problem be given

$$\begin{cases} u_t = & u_{xx} & \text{for} & 0 < t < \pi,\ 0 < t \le T \\ u(x,0) = & \sin x & \text{for} & 0 \le x \le \pi \\ u(0,t) = u(\pi,t) = & 0 & \text{for} & 0 \le t \le T \end{cases}$$

- Use product ansatz for the solution:

$$u(x,t) = q(t) \cdot p(x).$$

- Obtain two ordinary differential equations:

$$\begin{aligned} \dot{q}(t) + \delta q(t) &= 0, \\ p''(x) + \delta p(x) &= 0. \end{aligned}$$

- Solution classes Depending on δ:

$$\begin{aligned} u(x,t) &= c_0 e^{-\delta t} \cdot (c_1 x + c_2) \\ u(x,t) &= c_0 e^{-\delta t} \cdot (c_1 e^{-\sqrt{|\delta|}x} + c_2 e^{\sqrt{|\delta|}x}) \\ u(x,t) &= c_0 e^{-\delta t} \cdot (c_1 \sin(\sqrt{\delta}x) + c_2 \cos(\sqrt{\delta}x)) \end{aligned}$$

- Parameters $\{c_0, c_1, c_2, \delta\}$ generally cannot be determined by initial and boundary values allone.

Fundamental Solution of the Heat Equation

Definition: (Fundamental Solution to Heat Equation) The function

$$\Phi(\mathbf{x},t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & (\mathbf{x} \in \mathbb{R}^n, t > 0) \\ 0 & (\mathbf{x} \in \mathbb{R}^n, t < 0) \end{cases}$$

is called fundamental solution of the heat equations.

Remark: (Solution to the Cauchy Problem)
By means of $\Phi(\mathbf{x},t)$ the solution to the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times]0,\infty[\\ u = g & \text{on} \quad \mathbb{R}^n \times \{0\} \end{cases}$$

can be represented by a convolution integral:

$$\begin{aligned} u(\mathbf{x},t) &= \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t) g(\mathbf{y}) \, d\mathbf{y} \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} g(\mathbf{y}) \, d\mathbf{y}. \end{aligned}$$

3

Definition: (Fundamental Solution to Heat Equation) The function

$$\Phi(\mathbf{x},t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{|x|^2}{4t}} & (\mathbf{x} \in \mathbb{R}^n, t > 0) \\ 0 & (\mathbf{x} \in \mathbb{R}^n, t < 0) \end{cases}$$

is called fundamental solution of the heat equations.

Remarks:

- The fundamental solution is normalized, i.e. for all $t > 0$:

$$\int_{\mathbb{R}^n} \Phi(\mathbf{x},t)\, d\mathbf{x} = 1.$$

- The fundamental solution has singularities for $t = 0$ and $x = 0$.

Remark: (Solution to the Cauchy Problem)
By means of $\Phi(\mathbf{x}, t)$ the solution to the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times]0, \infty[\\ u = g & \text{on} \quad \mathbb{R}^n \times \{0\} \end{cases}$$

can be represented by a convolution integral:

$$\begin{aligned} u(\mathbf{x}, t) &= \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}, t) g(\mathbf{y}) \, d\mathbf{y} \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} g(\mathbf{y}) \, d\mathbf{y}. \end{aligned}$$

3

Representations of Solutions to the Heat Equation

The inhomogeneous initial value problem with homogeneous initial conditions

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } \mathbb{R}^n \times]0,\infty[\\ u(\mathbf{x},0) &= 0 \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

has the solution

$$\begin{aligned} u(\mathbf{x},t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y}, t-s) f(\mathbf{y},s)\, d\mathbf{y} ds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-s)}} f(\mathbf{y},s)\, d\mathbf{y} ds. \end{aligned}$$

The inhomogeneous initial value problem with inhomogeneous initial conditions

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } \mathbb{R}^n \times]0,\infty[\\ u(\mathbf{x},0) &= g(\mathbf{x}) \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

has the solution

$$u(\mathbf{x},t) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y}, t) g(\mathbf{y})\, d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y}, t-s) f(\mathbf{y},s)\, d\mathbf{y} ds.$$

Duhamel's Principle:
The function $u(\mathbf{x},t;s) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y}, t-s) f(\mathbf{y},s)\, d\mathbf{y}$ solves the problem

$$\begin{cases} u_t(\,\cdot\,;s) - \Delta u(\,\cdot\,;s) &= 0 \quad \text{in } \mathbb{R}^n \times]s,\infty[\\ u(\,\cdot\,;s) &= f(\,\cdot\,;s) \quad \text{on } \mathbb{R}^n \times \{t=s\} \end{cases}$$

One obtains the solution to the inhomogeneous problem by integrating over s:

$$u(\mathbf{x},t) = \int_0^t u(\mathbf{x},t;s)\, ds.$$

The inhomogeneous initial value problem with homogeneous initial conditions

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } \mathbb{R}^n \times]0, \infty[\\ u(\mathbf{x}, 0) &= 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

has the solution

$$\begin{aligned} u(\mathbf{x}, t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}, t - s) f(\mathbf{y}, s) \, d\mathbf{y} ds \\ &= \int_0^t \frac{1}{(4\pi(t - s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{x} - \mathbf{y}|^2}{4(t - s)}} f(\mathbf{y}, s) \, d\mathbf{y} ds. \end{aligned}$$

Duhamel's Principle:

The function $u(\mathbf{x},t;s) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t-s)f(\mathbf{y},s)\; d\mathbf{y}$ solves the problem

$$\begin{cases} u_t(\,\cdot\,;s) - \Delta u(\,\cdot\,;s) &= 0 \qquad\qquad \text{in } \mathbb{R}^n \times]s,\infty[\\ u(\,\cdot\,;s) &= f(\,\cdot\,;s) \quad \text{on } \mathbb{R}^n \times \{t=s\} \end{cases}$$

One obtains the solution to the inhomogeneous problem by integrating over s:

$$u(\mathbf{x},t) = \int_0^t u(\mathbf{x},t;s)\; ds.$$

The inhomogeneous initial value problem with inhomogeneous initial conditions

$$\begin{cases} u_t - \Delta u & = f & \text{in } \mathbb{R}^n \times]0, \infty[\\ u(\mathbf{x}, 0) & = g(\mathbf{x}) & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

has the solution

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}, t) g(\mathbf{y}) \; d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}, t - s) f(\mathbf{y}, s) \; d\mathbf{y} ds.$$

Properties of the Solution to the Heat Equation

Preliminary Remarks:

- **Goal:** Description of mean value forms.
- Let $U \subset \mathbb{R}^n$ open and bounded, $T > 0$ fixed. The set

$$U_T := U \times]0, T]$$

  is called parabolic cylinder with parabolic boundary $\Gamma_T := \overline{U_T} \setminus U_t$.
- For fixed $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $r > 0$ let the set $E(\mathbf{x}, t; r)$ be given by

$$E(\mathbf{x}, t; r) := \left\{ (\mathbf{y}, s) \in \mathbb{R}^{n+1} : s \leq t, \Phi(\mathbf{x} - \mathbf{y}, t - s) \geq \frac{1}{r^n} \right\}$$

- The boundary of $E(\mathbf{x}, t; r)$ is a contour line of the fundamental solution $\Phi(\mathbf{x} - \mathbf{y}, t - s)$. $E(\mathbf{x}, t; r)$ is called heat ball.

Proposition: (Mean Value Property of Heat Equation)
If $u \in C_1^2(U_t)$ is a solution of the heat equation, then

$$u(\mathbf{x}, t) = \frac{1}{4r^n} \int_{E(\mathbf{x}, t; r)} \frac{|\mathbf{x} - \mathbf{y}|^2}{(t - s)^2} u(\mathbf{y}, s) \, d\mathbf{y} ds$$

for each set $E(\mathbf{x}, t; r) \subset U_t$.

Remark:
One can show that for the Cauchy problem

$$\begin{cases} u_t &= \Delta u \quad \text{in } \mathbb{R}^n \times]0, T[\\ u &= 0 \quad \text{auf } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

there exist indefinitely many solutions.
Only the Null solution fulfills the growth condition,
all other solutions grow rapidly.

Example:
Numerical Solution of Heat Equation

Proposition: (Solution to the Cauchy Problem under Growth Conditions)
The initial value problem

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } \mathbb{R}^n \times]0, T[\\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

on the whole $\mathbb{R}^n$ with continuous functions f and g and with the additional growth condition

$$|u(\mathbf{x}, t)| \leq A e^{a|\mathbf{x}|^2} \quad (A, a > 0)$$

has at most one solution $u \in C_1^2(\mathbb{R}^n \times]0, T]) \cap C(\mathbb{R}^n \times [0, T])$.

Proposition: (Maximum Principles of Heat Equation)
Let $u \in C_1^2(U_t) \cap C(\overline{U_T})$ be a solution to the heat equations in U_T. Then it holds:

1. The maximum of $u(\mathbf{x}, t)$ is always on the parabolic boundary, i.e.

$$\max_{(\mathbf{x}, t) \in \overline{U_T}} u(\mathbf{x}, t) = \max_{(\mathbf{x}, t) \in \Gamma_T} u(\mathbf{x}, t)$$

2. If U is connected and if a point $(\mathbf{x}_0, t_0) \in U_T$ exists with

$$u(\mathbf{x}_0, t_0) = \max_{(\mathbf{x}, t) \in \overline{U_T}} u(\mathbf{x}, t),$$

   then u is constant in U_T.

Proposition: (Unique Solution of Heat Equation)
The initial value problem

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } U_t \\ u &= g \quad \text{auf } \Gamma_T \end{cases}$$

on the bounded domain U_T with continuous functions f and g
has at most one solution $u \in C_1^2(U_T) \cap C(\overline{U_T})$.

Preliminary Remarks:

- **Goal:** Description of mean value forms.
- Let $U \subset \mathbb{R}^n$ open and bounded, $T > 0$ fixed. The set

$$U_T := U \times]0, T]$$

is called parabolic cylinder with parabolic boundary $\Gamma_T := \overline{U_T} \backslash U_t$.

- For fixed $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $r > 0$ let the set $E(\mathbf{x}, t; r)$ be given by

$$E(\mathbf{x}, t; r) := \left\{ (\mathbf{y}, s) \in \mathbb{R}^{n+1} : s \leq t, \Phi(\mathbf{x} - \mathbf{y}, t - s) \geq \frac{1}{r^n} \right\}$$

- The boundary of $E(\mathbf{x}, t; r)$ is a contour line of the fundamental solution $\Phi(\mathbf{x} - \mathbf{y}, t - s)$. $E(\mathbf{x}, t; r)$ is called heat ball.

Proposition: (Mean Value Property of Heat Equation)
If $u \in C_1^2(U_t)$ is a solution of the heat equation, then

$$u(\mathbf{x}, t) = \frac{1}{4r^n} \int_{E(\mathbf{x},t;r)} \frac{|\mathbf{x} - \mathbf{y}|^2}{(t-s)^2} u(\mathbf{y}, s) \; d\mathbf{y} ds$$

for each set $E(\mathbf{x}, t; r) \subset U_t$.

Proposition: (Maximum Principles of Heat Equation)
Let $u \in C_1^2(U_t) \cap C(\overline{U_T})$ be a solution to the heat equations in U_T. Then it holds:

1. The maximum of $u(\mathbf{x}, t)$ is always on the parabolic boundary, i.e.

$$\max_{(\mathbf{x},t)\in\overline{U_T}} u(\mathbf{x}, t) = \max_{(\mathbf{x},t)\in\Gamma_T} u(\mathbf{x}, t)$$

2. If U is connected and if a point $(\mathbf{x}_0, t_0) \in U_T$ exists with

$$u(\mathbf{x}_0, t_0) = \max_{(\mathbf{x},t)\in\overline{U_T}} u(\mathbf{x}, t),$$

then u is constant in U_T.

Proposition: (Unique Solution of Heat Equation)
The initial value problem

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } U_t \\ u &= g \quad \text{auf } \Gamma_T \end{cases}$$

on the bounded domain U_T with continuous functions f and g has at most one solution $u \in C_1^2(U_T) \cap C(\overline{U_T})$.

Proof:
If u and $\tilde{u}$ are two solutions, then the two functions

$$w_{1/2} = \pm(u - \tilde{u})$$

solve the homogeneous heat equation with homogeneous boundary conditions.
According to the maximum principle $w_{1/2}$ then vanishes, i.e. $u = \tilde{u}$.

Proposition: (Solution to the Cauchy Problem under Growth Conditions)
The initial value problem

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } \mathbb{R}^n \times]0,T[\\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

on the whole $\mathbb{R}^n$ with continuous functions f and g and with the additional growth condition

$$|u(\mathbf{x},t)| \leq Ae^{a|\mathbf{x}|^2} \quad (A, a > 0)$$

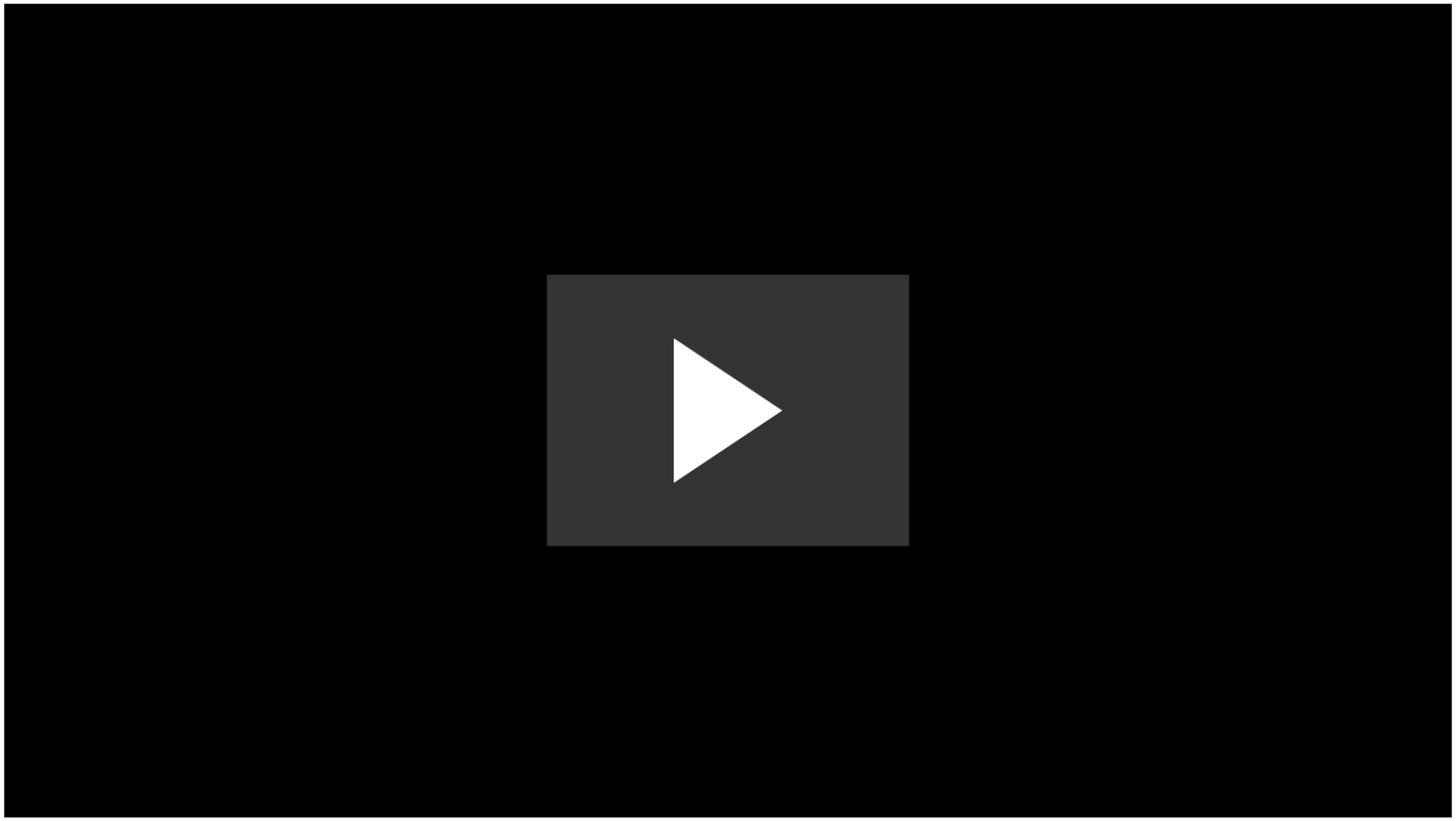
has at most one solution $u \in C^2_1(\mathbb{R}^n \times]0,T[) \cap C(\mathbb{R}^n \times [0,T])$.

Remark:
One can show that for the Cauchy problem

$$\begin{cases} u_t &= \Delta u \quad \text{in } \mathbb{R}^n \times]0,T[\\ u &= 0 \qquad \text{auf } \mathbb{R}^n \times \{t=0\} \end{cases}$$

there exist indefinitely many solutions.
Only the Null solution fulfills the growth condition,
all other solutions grow rapidly.

Example:
Numerical Solution of Heat Equation




Fundamental Solution of the Heat Equation
Heat Equation
Differential Equations II
Representations of Solutions to the Heat Equation
Properties of the Solution to the Heat Equation