

Differential Equations II



Fourier Methods

Motivating Example in 1D

Preliminary Remark:

Consider the one-dimensional boundary value problem (Poisson's Equation)

$$\begin{cases} -T \frac{d^2 u}{dx^2} = f(x), & 0 < x < l \\ u(0) = u(l) = 0 \end{cases}$$

Application: The equation describes the equilibrium position of a fixed hanging rope with tension T and external force $f(x)$.

1

Remark: (General Approximate Solution of 1D Poisson's Equation)

- Let the one-dimensional boundary value problem be given:

$$\begin{cases} -T \frac{d^2 u}{dx^2} = f(x), & 0 < x < l \\ u(0) = u(l) = 0 \end{cases}$$

- Approximate the right hand side $f(x)$ by a finite Fourier series $f_N(x)$:

$$f_N(x) = \sum_{n=1}^N c_n \sin\left(\frac{n\pi x}{l}\right).$$

- The Fourier coefficients are $(n = 1, \dots, N)$

$$c_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

- Then an approximate solution of the boundary value problem is given by:

$$u_N(x) = \sum_{n=1}^N \frac{l^2 c_n}{T n^2 \pi^2} \sin\left(\frac{n\pi x}{l}\right).$$

2

Preliminary Remark:

Consider the one-dimensional boundary value problem (**Poisson's Equation**)

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2

Fourier Method for the Heat Equation

Remember: (Heat Equation) Consider the initial boundary value problem of the heat equation:

$$\begin{cases} u_t - u_{xx} = f(x,t) & ; 0 < x < l, 0 < t \leq T, \\ u(x,0) = g(x) & ; 0 \leq x \leq l, \\ u(0,t) = u(l,t) = 0 & ; 0 \leq t \leq T. \end{cases}$$

We look for a solution in form of a Fourier series:

$$u_N(x) = \sum_{n=1}^N a_n(t) \sin\left(\frac{n\pi x}{l}\right).$$

Remark: Since only \sin appears in the Fourier series, homogeneous boundary conditions are automatically satisfied.

Linear System of Decoupled ODEs:

- Obtain a decoupled linear system of ODEs:

$$a_n' + k_n a_n = c_n, \quad n = 1, 2, \dots$$

with $k_n = \frac{n^2 \pi^2}{l^2}$.

- The solutions can be given directly:

$$a_n(t) = b_n \exp(-k_n t) + \int_0^t \exp(-k_n(t-s)) c_n(s) ds.$$

3

Solution Approach:

- For the coefficients of the solution representation we have:

$$a_n(t) = \frac{2}{l} \int_0^l u(x,t) \sin\left(\frac{n\pi x}{l}\right) dx.$$

- The right hand side (inhomogeneity) $f(x,t)$ can be written

$$f(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad \text{with } c_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin\left(\frac{n\pi x}{l}\right) dx.$$

- Compute temporal and spatial derivatives of the solution approach for u :

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \sum_{n=1}^{\infty} \frac{\partial a_n(t)}{\partial t} \sin\left(\frac{n\pi x}{l}\right) \\ \frac{\partial u}{\partial x}(x,t) &= \sum_{n=1}^{\infty} a_n(t) \frac{\partial}{\partial x} \sin\left(\frac{n\pi x}{l}\right) \\ \frac{\partial^2 u}{\partial x^2}(x,t) &= \sum_{n=1}^{\infty} a_n(t) \frac{\partial^2}{\partial x^2} \sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

Comparison of Coefficients:

- Obtain for left hand side of heat equation

$$u_t + u_{xx} = \sum_{n=1}^{\infty} \left(\frac{\partial a_n(t)}{\partial t} + a_n(t) \frac{\partial^2}{\partial x^2} \right) \sin\left(\frac{n\pi x}{l}\right).$$

- Comparison of coefficients with Fourier series for $f(x,t)$ yields a system of ODEs:

$$\frac{\partial a_n(t)}{\partial t} + a_n(t) \frac{n^2 \pi^2}{l^2} = c_n(t).$$

- Coefficients $a_1(0), a_2(0), \dots$ are obtained from initial conditions $u(x,0) = g(x)$:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \quad b_n = \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\Rightarrow a_n(0) = b_n, \quad n = 1, 2, \dots$$

Remember: (Heat Equation) Consider the initial boundary value problem of the heat equation:

$$\begin{cases} u_t - u_{xx} = f(x, t) & : 0 < x < l, 0 < t \leq T, \\ u(x, 0) = g(x) & : 0 \leq x \leq l, \\ u(0, t) = u(l, t) = 0 & : 0 \leq t \leq T. \end{cases}$$

We look for a solution in form of a Fourier series:

$$u_N(x) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right).$$

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$$f(x, t) = \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{n\pi x}{l}\right), \quad \text{with } c_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin\left(\frac{n\pi x}{l}\right) dx.$$

- Compute temporal and spatial derivatives of the solution approach for u :

$$\frac{\partial u}{\partial t}(x, t) = \sum_{n=1}^{\infty} \frac{\partial a_n}{\partial t}(t) \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{\partial u}{\partial x}(x, t) = \sum_{n=1}^{\infty} a_n(t) \frac{n\pi}{l} \cos\left(\frac{n\pi x}{l}\right)$$

$$\frac{\partial^2 u}{\partial x^2}(x, t) = - \sum_{n=1}^{\infty} a_n(t) \frac{n^2 \pi^2}{l^2} \sin\left(\frac{n\pi x}{l}\right)$$

Comparison of Coefficients:

- Obtain for left hand side of heat equation

$$u_t + u_{xx} = \sum_{n=1}^{\infty} \left(\frac{\partial a_n}{\partial t}(t) + a_n(t) \frac{n^2 \pi^2}{l^2} \right) \sin \left(\frac{n\pi x}{l} \right).$$

- Comparison of coefficients with Fourier series for $f(x, t)$ yields a system of ODEs:

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- Coefficients $a_1(0), a_2(0), \dots$ are obtained from initial conditions $u(x, 0) = g(x)$:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l} \right), \quad b_n = \frac{2}{l} \int_0^l g(x) \sin \left(\frac{n\pi x}{l} \right) dx$$
$$\Rightarrow a_n(0) = b_n, \quad n = 1, 2, \dots$$

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- Obtain a decoupled linear system of ODEs:

$$\dot{a}_n + k_n a_n = c_n, \quad n = 1, 2, \dots$$

with $k_n = \frac{n^2 \pi^2}{l^2}$.

- The solutions can be given directly:

$$a_n(t) = b_n \exp(-k_n \cdot t) + \int_0^t \exp(-k_n \cdot (t - s)) c_n(s) ds.$$

3

Fourier Method: Properties, Boundary Conditions

Observation:

- For $T > 0$ fixed, the $a_n(t)$ decay exponentially fast ($n \rightarrow \infty$).
Higher values for n represent higher frequencies in the solution.
- For n fixed, the $a_n(t)$ decay exponentially fast ($t \rightarrow \infty$).
The decay is faster, the larger n . For t large only few terms in the Fourier series suffice for an accurate solution.

Periodic Boundary Conditions

Let the initial boundary value problem be given on interval $[-L, L]$:

$$\begin{cases} u_t - u_{xx} = f(x, t) & : -L < x < L, 0 < t \leq T, \\ u(x, 0) = g(x) & : -L \leq x \leq L, \\ u(-L, t) = u(L, t) & : 0 \leq t \leq T, \\ u_x(-L, t) = u_x(L, t) & : 0 \leq t \leq T. \end{cases}$$

Periodic functions on $[-L, L]$ are

$$u(x) = \frac{1}{2} \left[u(x) + u\left(\frac{2L-x}{T}\right) \right], \quad u(x) = \cos\left(\frac{n\pi x}{L}\right)$$

satisfy the given Neumann boundary conditions.

A solution approach using Fourier series thus reads

$$u(x, t) = u_0(t) + \sum_{n=1}^{\infty} \left(a_n(t) \cos\left(\frac{n\pi x}{L}\right) + b_n(t) \sin\left(\frac{n\pi x}{L}\right) \right)$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{d a_n(t)}{dt} \cos\left(\frac{n\pi x}{L}\right) + \frac{d b_n(t)}{dt} \sin\left(\frac{n\pi x}{L}\right) \right) \\ & - \sum_{n=1}^{\infty} \left(-\frac{n^2 \pi^2}{L^2} a_n(t) \cos\left(\frac{n\pi x}{L}\right) - \frac{n^2 \pi^2}{L^2} b_n(t) \sin\left(\frac{n\pi x}{L}\right) \right) \\ & = f(x, t) \end{aligned}$$

Example:

Consider the inhomogeneous initial boundary value problem

$$\begin{cases} u_t - u_{xx} = x & : 0 < x < 1, 0 < t \leq T, \\ u(x, 0) = 0 & : 0 \leq x \leq 1, \\ u(0, t) = u(1, t) = 0 & : 0 \leq t \leq T. \end{cases}$$

With the previous results we have:

$$b_n = 0$$

$$a_n(t) = a_n = 2 \frac{(-1)^{n+1}}{n^3 \pi^3}$$

And therefore

$$u(x, t) = 2 \int_0^t e^{-x^2(x-y)} \left(\frac{(-1)^{n+1}}{n^3 \pi^3} \right) dx = 2 \frac{(-1)^{n+1}}{n^3 \pi^3} \left(1 - e^{-x^2 t} \right)$$

So far:

Initial boundary value problems with homogeneous boundary conditions, i.e.

$$\begin{cases} u_t - u_{xx} = f(x, t) & : 0 < x < L, 0 < t \leq T, \\ u(x, 0) = g(x) & : 0 \leq x \leq L, \\ u(0, t) = u(L, t) = 0 & : 0 \leq t \leq T. \end{cases}$$

If both sides are (heat) isolated, then we obtain the initial boundary value problem

$$\begin{cases} u_t - u_{xx} = f(x, t) & : 0 < x < L, 0 < t \leq T, \\ u(x, 0) = g(x) & : 0 \leq x \leq L, \\ u_x(0, t) = 0 & : 0 \leq t \leq T, \\ u_x(L, t) = 0 & : 0 \leq t \leq T. \end{cases}$$

Observation:

- For $T > 0$ fixed, the $a_n(t)$ decay exponentially fast ($n \rightarrow \infty$).
Higher values for n represent higher frequencies in the solution.
- For n fixed, the $a_n(t)$ decay exponentially fast ($t \rightarrow \infty$).
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And therefore

$$a_n(t) = 2 \int_0^t e^{-n^2 \pi^2 (t-s)} \frac{(-1)^{n+1}}{n\pi} ds = 2 \frac{(-1)^{n+1}}{n^3 \pi^3} \left(1 - e^{-n^2 \pi^2 t} \right).$$

So far:

Initial boundary value problems with homogeneous boundary conditions, i.e.

$$\begin{cases} u_t - u_{xx} = f(x, t) & : 0 < x < l, 0 < t \leq T, \\ u(x, 0) = g(x) & : 0 \leq x \leq l, \\ u(0, t) = u(l, t) = 0 & : 0 \leq t \leq T. \end{cases}$$

Question:

What if

1. (onesided) **Neumann** boundary conditions

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(l, t) = 0;$$

2. **periodic** boundary conditions

$$u(0, t) = u(l, t), \quad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(l, t).$$

How do the corresponding Fourier methods look like?

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If both sides are **(heat) isolated**, then we obtain the initial boundary value problem

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In this case the functions

$$u(x, t) = 1, \quad u(x, t) = \cos\left(\frac{\pi x}{l}\right)$$

satisfy the given Neumann boundary conditions.

A **solution approach** therefore reads

$$u(x, t) = b_0(t) + \sum_{n=1}^{\infty} b_n(t) \cos\left(\frac{n\pi x}{l}\right)$$

Periodic Boundary Conditions

Let the initial boundary value problem be given on **interval** $[-l, l]$:

$$\begin{cases} u_t - u_{xx} = f(x, t) & : -l < x < l, 0 < t \leq T, \\ u(x, 0) = g(x) & : -l \leq x \leq l, \\ u(-l, t) = u(l, t) & : 0 \leq t \leq T, \\ u_x(-l, t) = u_x(l, t) & : 0 \leq t \leq T. \end{cases}$$

Periodic functions on $[-l, l]$ are

$$\psi(x) = \frac{1}{2}, \quad \psi(x) = \cos\left(\frac{n\pi x}{l}\right), \quad \psi(x) = \sin\left(\frac{n\pi x}{l}\right)$$

satisfy the given Neumann boundary conditions.

A solution approach using Fourier series thus reads

$$u(x, t) = a_0(t) + \sum_{n=1}^{\infty} \left(a_n(t) \cos\left(\frac{n\pi x}{l}\right) + b_n(t) \sin\left(\frac{n\pi x}{l}\right) \right).$$

With series expansions

$$f(x, t) = c_0(t) + \sum_{n=1}^{\infty} \left(c_n(t) \cos\left(\frac{n\pi x}{l}\right) + d_n(t) \sin\left(\frac{n\pi x}{l}\right) \right)$$

$$g(x) = p_0 + \sum_{n=1}^{\infty} \left(p_n \cos\left(\frac{n\pi x}{l}\right) + q_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

we obtain the ordinary differential equations

$$\begin{aligned} \frac{da_0}{dt}(t) &= c_0(t) \\ \frac{da_n}{dt}(t) + \frac{n^2\pi^2}{l^2}a_n(t) &= c_n(t) \\ \frac{db_n}{dt}(t) + \frac{n^2\pi^2}{l^2}b_n(t) &= d_n(t) \end{aligned}$$

with corresponding initial conditions

$$a_0(0) = p_0, \quad a_n(0) = p_n, \quad b_n(0) = q_n$$

Fourier Method for the Wave Equation

Idea: Seek solution analogously to heat equation

Consider the initial boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & : 0 < x < l, 0 < t \leq T, \\ u(x,0) = g(x) & : 0 \leq x \leq l, \\ u_t(x,0) = h(x) & : 0 \leq x \leq l, \\ u(0,t) = u(l,t) = 0 & : 0 \leq t \leq T. \end{cases}$$

Now, seek a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right).$$

The Fourier series expansions for $f(x,t)$, $g(x)$, and $h(x)$ yield ODEs for the coefficients $a_i(t)$, $i = 1, 2, \dots$

Example:

The solution for the initial boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & : 0 < x < l, 0 < t \leq T, \\ u(x,0) = g(x) & : 0 \leq x \leq l, \\ u_t(x,0) = h(x) & : 0 \leq x \leq l, \\ u(0,t) = u(l,t) = 0 & : 0 \leq t \leq T. \end{cases}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[b_n \cos\left(\frac{n\pi t}{T}\right) + \frac{d_n}{n\pi} \sin\left(\frac{n\pi t}{T}\right) \right] \sin\left(\frac{n\pi x}{l}\right)$$

where b_n are the Fourier coefficients corresponding to the series expansion for initial conditions $u(x,0) = g(x)$ and d_n the corresponding coefficients for condition $u_t(x,0) = h(x)$.

Idea: Seek solution analogously to heat equation

Consider the initial boundary value problem

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The Fourier series expansions for $f(x, t)$, $g(x)$, and $h(x)$ yield ODEs for the coefficients $a_i(t)$, $i = 1, 2, \dots$

Example:

The solution for the initial boundary value problem

$$\left\{ \begin{array}{ll} u_{tt} - u_{xx} = 0 & : 0 < x < l, 0 < t \leq T, \\ u(x, 0) = g(x) & : 0 \leq x \leq l, \\ u_t(x, 0) = h(x) & : 0 \leq x \leq l, \\ u(0, t) = u(l, t) = 0 & : 0 \leq t \leq T. \end{array} \right.$$

is given by

$$u(x, t) = \sum_{n=1}^{\infty} \left[b_n \cos\left(\frac{n\pi}{l}t\right) + \frac{d_n l}{n\pi} \sin\left(\frac{n\pi}{l}t\right) \right] \sin\left(\frac{n\pi x}{l}\right)$$

where b_n are the Fourier coefficients corresponding to the series expansion for initial conditions $u(x, 0) = g(x)$ and d_n the corresponding coefficients for condition $u_t(x, 0) = h(x)$.

Motivating Example in 1D

Problem Statement
Solve the 1D heat conduction problem (1D heat equation):
$$u_t = \alpha u_{xx}, \quad 0 < x < L, \quad t > 0$$

Boundary conditions: $u(0, t) = u(L, t) = 0$
Initial condition: $u(x, 0) = f(x)$

Method: Separation of Variables
Assume $u(x, t) = X(x)T(t)$
Substitute into the PDE:
 $X(x)T'(t) = \alpha X''(x)T(t)$
Divide by $X(x)T(t)$:
 $\frac{T'(t)}{T(t)} = \alpha \frac{X''(x)}{X(x)}$
Set both sides equal to a constant $-\lambda$:
 $\frac{T'(t)}{T(t)} = -\lambda T(t) \implies T(t) = e^{-\lambda t}$
 $\alpha \frac{X''(x)}{X(x)} = -\lambda X(x) \implies X''(x) + \frac{\lambda}{\alpha} X(x) = 0$
Boundary conditions: $X(0) = X(L) = 0$
Eigenvalues: $\lambda_n = \alpha \left(\frac{n\pi}{L}\right)^2$
Eigenfunctions: $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$
General solution:
$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n t}$$

Fourier Method for the Heat Equation

Problem Statement
Solve the 1D heat conduction problem (1D heat equation):
$$u_t = \alpha u_{xx}, \quad 0 < x < L, \quad t > 0$$

Boundary conditions: $u(0, t) = u(L, t) = 0$
Initial condition: $u(x, 0) = f(x)$

Method: Separation of Variables
Assume $u(x, t) = X(x)T(t)$
Substitute into the PDE:
 $X(x)T'(t) = \alpha X''(x)T(t)$
Divide by $X(x)T(t)$:
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$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n t}$$

Fourier Method: Properties, Boundary Conditions

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Solve the 1D heat conduction problem (1D heat equation):
$$u_t = \alpha u_{xx}, \quad 0 < x < L, \quad t > 0$$

Boundary conditions: $u(0, t) = u(L, t) = 0$
Initial condition: $u(x, 0) = f(x)$

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Substitute into the PDE:
 $X(x)T'(t) = \alpha X''(x)T(t)$
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 $\frac{T'(t)}{T(t)} = \alpha \frac{X''(x)}{X(x)}$
Set both sides equal to a constant $-\lambda$:
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Eigenfunctions: $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$
General solution:
$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n t}$$

Fourier Method for the Wave Equation

Problem Statement
Solve the 1D wave equation:
$$u_{tt} = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0$$

Boundary conditions: $u(0, t) = u(L, t) = 0$
Initial conditions: $u(x, 0) = f(x), u_t(x, 0) = g(x)$

Method: Separation of Variables
Assume $u(x, t) = X(x)T(t)$
Substitute into the PDE:
 $X(x)T''(t) = \alpha^2 X''(x)T(t)$
Divide by $X(x)T(t)$:
 $\frac{T''(t)}{T(t)} = \alpha^2 \frac{X''(x)}{X(x)}$
Set both sides equal to a constant $-\lambda$:
 $\frac{T''(t)}{T(t)} = -\lambda T(t) \implies T(t) = \cos(\sqrt{\lambda} t) \text{ or } \sin(\sqrt{\lambda} t)$
 $\alpha^2 \frac{X''(x)}{X(x)} = -\lambda X(x) \implies X''(x) + \frac{\lambda}{\alpha^2} X(x) = 0$
Boundary conditions: $X(0) = X(L) = 0$
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General solution:
$$u(x, t) = \sum_{n=1}^{\infty} \left[b_n \cos\left(\frac{n\pi \alpha t}{L}\right) + c_n \sin\left(\frac{n\pi \alpha t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$