Differential Equations II

Fourier Methods

Motivating Example in 1D

Preliminary Remark:
Consider the one-dimensional boundary value problem (Poisson's Equation)

$$\begin{cases} -T\frac{d^2u}{dx^2} = f(x), & 0 < x < l \\ u(0) = u(l) = 0 \end{cases}$$

Application: The equation describes the equilibrium position of a fixed hanging rope with tension T and external force $f(x)$.

1

Remark: (General Approximate Solution of 1D Poisson's Equation)

- Let the one-dimensional boundary value problem be given:

$$\begin{cases} -T\frac{d^2u}{dx^2} = f(x), & 0 < x < l, \\ u(0) = u(l) = 0. \end{cases}$$

- Approximate the right hand side $f(x)$ by a finite Fourier series $f_N(x)$:

$$f_N(x) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{l}\right).$$

- The Fourier coefficients are ($n = 1, \ldots, N$)

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\, dx.$$

- Then an approximate solution of the boundary value problem is given by:

$$u_N(x) = \sum_{n=1}^{N} \frac{l^2 c_n}{T n^2 \pi^2}\sin\left(\frac{n\pi x}{l}\right).$$

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Consider the one-dimensional boundary value problem (Poisson's Equation)

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2

Fourier Method for the Heat Equation

Remember: (Heat Equation) Consider the initial boundary value problem of the heat equation:

$$\begin{cases} u_t - u_{xx} = f(x,t) & : 0 < x < l,\ 0 < t \le T, \\ u(x,0) = g(x) & : 0 \le x \le l, \\ u(0,t) = u(l,t) = 0 & : 0 \le t \le T. \end{cases}$$

We look for a solution in form of a Fourier series:

$$u_N(x) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right).$$

Remark: Since only sin appears in the Fourier series, homogeneous boundary conditions are automatically satisfied.

Solution Approach:

- For the coefficients of the solution representation we have:

$$a_n(t) = \frac{2}{l}\int_0^l u(x,t)\sin\left(\frac{n\pi x}{l}\right)\,dx.$$

- The right hand side (Inhomogeneity) $f(x,t)$ can be written

$$f(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin\left(\frac{n\pi x}{l}\right), \quad \text{with } c_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\left(\frac{n\pi x}{l}\right)\,dx.$$

- Compute temporal and spatial derivatives of the solution approach for u:

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \sum_{n=1}^{\infty} \frac{\partial a_n}{\partial t}(t)\sin\left(\frac{n\pi x}{l}\right) \\ \frac{\partial u}{\partial x}(x,t) &= \sum_{n=1}^{\infty} a_n(t)\frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right) \\ \frac{\partial^2 u}{\partial x^2}(x,t) &= -\sum_{n=1}^{\infty} a_n(t)\frac{n^2\pi^2}{l^2}\sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

Comparison of Coefficients:

- Obtain for left hand side of heat equation

$$u_t + u_{xx} = \sum_{n=1}^{\infty}\left(\frac{\partial a_n}{\partial t}(t) + a_n(t)\frac{n^2\pi^2}{l^2}\right)\sin\left(\frac{n\pi x}{l}\right).$$

- Comparison of coefficients with Fourier series for $f(x,t)$ yields a system of ODEs:

$$\frac{\partial a_n}{\partial t}(t) + a_n(t)\frac{n^2\pi^2}{l^2} = c_n(t).$$

- Coefficients $a_1(0), a_2(0), \dots$ are obtained from initial conditions $u(x,0) = g(x)$:

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \quad b_n = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \\ \Rightarrow\quad a_n(0) &= b_n, \quad n = 1, 2, \dots \end{aligned}$$

Linear System of Decoupled ODEs:

- Obtain a decoupled linear system of ODEs:

$$\dot{a}_n + k_n a_n = c_n, \quad n = 1, 2, \dots$$

with $k_n = \frac{n^2\pi^2}{l^2}$.

- The solutions can be given directly:

$$a_n(t) = b_n \exp\left(-k_n \cdot t\right) + \int_0^t \exp\left(-k_n \cdot (t-s)\right) c_n(s)\,ds.$$

Remember: (Heat Equation) Consider the initial boundary value problem of the heat equation:

$$\begin{cases} u_t - u_{xx} = f(x,t) & : \quad 0 < x < l, \; 0 < t \leq T, \\ u(x,0) = g(x) & : \quad 0 \leq x \leq l, \\ u(0,t) = u(l,t) = 0 & : \quad 0 \leq t \leq T. \end{cases}$$

We look for a solution in form of a Fourier series:

$$u_N(x) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right).$$

Remark: Since only $\sin$ appears in the Fourier series, homogeneous boundary conditions are automatically satisfied.

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$$\begin{aligned}
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\end{aligned}$$

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Fourier Method:
Properties, Boundary Conditions

Observation:

- For $T > 0$ fixed, the $a_n(t)$ decay exponentially fast $(n \to \infty)$.
  Higher values for n represent higher frequencies in the solution.
- For n fixed, the $a_n(t)$ decay exponentially fast $(t \to \infty)$.
  The decay is faster, the larger n. For t large only few terms in the Fourier series suffice for an accurate solution.

Example:
Consider the inhomogeneous initial boundary value problem

$$\begin{cases} u_t - u_{xx} = x & : 0 < x < 1,\ 0 < t \le T, \\ u(x,0) = 0 & : 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0 & : 0 \le t \le T. \end{cases}$$

The with the previous results we have:

$$b_n = 0$$

$$c_n(t) = c_n = 2\frac{(-1)^{n+1}}{n\pi}$$

And therefore

$$a_n(t) = 2\int_0^t e^{-n^2\pi^2(t-s)}\frac{(-1)^{n+1}}{n\pi}\,ds = 2\frac{(-1)^{n+1}}{n^3\pi^3}\left(1 - e^{-n^2\pi^2 t}\right).$$

Periodic Boundary Conditions

Let the initial boundary value problem be given on **interval** $[-l, l]$:

$$\begin{cases} u_t - u_{xx} = f(x,t) & : -l < x < l,\ 0 < t \le T, \\ u(x,0) = g(x) & : -l \le x \le l, \\ u(-l,t) = u(l,t) & : 0 \le t \le T, \\ u_x(-l,t) = u_x(l,t) & : 0 \le t \le T. \end{cases}$$

Periodic functions on $[-l, l]$ are

$$\psi(x) = \frac{1}{2}, \quad \psi(x) = \cos\left(\frac{n\pi x}{l}\right), \quad \phi(x) = \sin\left(\frac{n\pi x}{l}\right)$$

satisfy the given Neumann boundary conditions.
A **solution approach** using Fourier series thus reads

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty}\left(a_n(t)\cos\left(\frac{n\pi x}{l}\right) + b_n(t)\sin\left(\frac{n\pi x}{l}\right)\right).$$

So far:
Initial boundary value problems with homogeneous boundary conditions, i.e.

$$\begin{cases} u_t - u_{xx} = f(x,t) & : 0 < x < l,\ 0 < t \le T, \\ u(x,0) = g(x) & : 0 \le x \le l, \\ u(0,t) = u(l,t) = 0 & : 0 \le t \le T. \end{cases}$$

If both sides are **(heat) isolated**, then we obtain the initial boundary value problem

$$\begin{cases} u_t - u_{xx} = f(x,t) & : 0 < x < l,\ 0 < t \le T, \\ u(x,0) = g(x) & : 0 \le x \le l, \\ u_x(0,t) = 0 & : 0 \le t \le T, \\ u_x(l,t) = 0 & : 0 \le t \le T. \end{cases}$$

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Question:

What if

1. (onesided) **Neumann** boundary conditions

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(l,t) = 0;$$

2. **periodic** boundary conditions

$$u(0,t) = u(l,t), \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t).$$

How do the corresponding Fourier methods look like?

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In this case the functions

$$u(x,t) = 1, \quad u(x,t) = \cos\left(\frac{\pi x}{l}\right)$$

satisfy the given Neumann boundary conditions.

A **solution approach** therefore reads

$$u(x,t) = b_0(t) + \sum_{n=1}^{\infty} b_n(t) \cos\left(\frac{n\pi x}{l}\right)$$

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$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} \left(a_n(t) \cos\left(\frac{n\pi x}{l}\right) + b_n(t) \sin\left(\frac{n\pi x}{l}\right) \right).$$

With series expansions

$$f(x,t) = c_0(t) + \sum_{n=1}^{\infty} \left(c_n(t) \cos\left(\frac{n\pi x}{l}\right) + d_n(t) \sin\left(\frac{n\pi x}{l}\right) \right)$$

$$g(x) = p_0 + \sum_{n=1}^{\infty} \left(p_n \cos\left(\frac{n\pi x}{l}\right) + q_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

we obtain the ordinary differential equations

$$\frac{da_0}{dt}(t) = c_0(t)$$

$$\frac{da_n}{dt}(t) + \frac{n^2\pi^2}{l^2} a_n(t) = c_n(t)$$

$$\frac{db_n}{dt}(t) + \frac{n^2\pi^2}{l^2} b_n(t) = d_n(t)$$

with corresponding initial conditions

$$a_0(0) = p_0, \quad a_n(0) = p_n, \quad b_n(0) = q_n$$

Fourier Method for the Wave Equation

Idea: Seek solution analogously to heat equation

Consider the initial boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & : \ 0 < x < l,\ 0 < t \le T, \\ u(x,0) = g(x) & : \ 0 \le x \le l, \\ u_t(x,0) = h(x) & : \ 0 \le x \le l, \\ u(0,t) = u(l,t) = 0 & : \ 0 \le t \le T. \end{cases}$$

Now, seek a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right).$$

The Fourier series expansions for $f(x,t)$, $g(x)$, and $h(x)$ yield ODEs for the coefficients $a_i(t)$, $i = 1, 2, \ldots$

Example:
The solution for the initial boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & : \ 0 < x < l,\ 0 < t \le T, \\ u(x,0) = g(x) & : \ 0 \le x \le l, \\ u_t(x,0) = h(x) & : \ 0 \le x \le l, \\ u(0,t) = u(l,t) = 0 & : \ 0 \le t \le T. \end{cases}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[b_n \cos\left(\frac{n\pi}{l} t\right) + \frac{d_n l}{n\pi} \sin\left(\frac{n\pi}{l} t\right) \right] \sin\left(\frac{n\pi x}{l}\right)$$

where b_n are the Fourier coefficients corresponding to the series expansion for initial conditions $u(x,0) = g(x)$ and d_n the corresponding coefficients for condition $u_t(x,0) = h(x)$.

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