Differential Equations II

Introduction to Numerical Methods

Preliminary Remarks

Classes of methods used for numerically solving PDEs:

- **Finite Differences**
  - Simplification of differential operator, all equation types, simple geometries and structured uniform meshes
- **Finite Volumes**
  - Simplification of physical principle, mostly hyperbolic equations
- **Finite Elements**
  - Simplification of function spaces, mostly elliptic equations, complex geometries and differential operators
- **Lagrangian Methods**
  - Simplification of material derivative, transport equation
- **Combinations** of those methods

Consider: Elliptic Model Problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega = [0,1]^2 \in \mathbb{R}^2 \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

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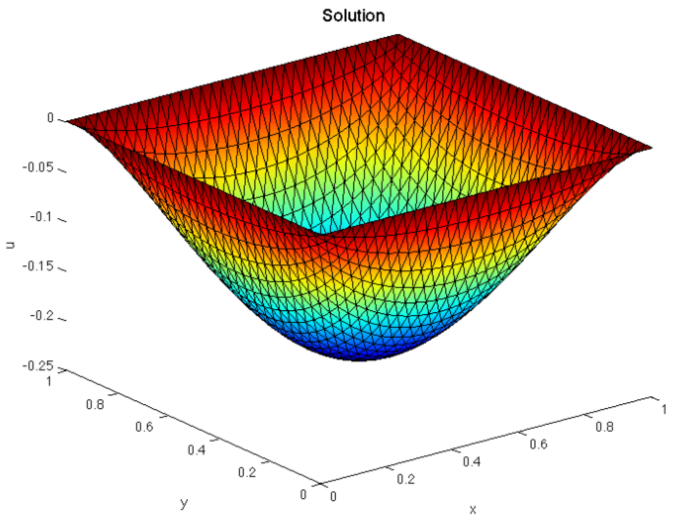
$$



Finite Differences

Idea:

Discretize the differencial operator

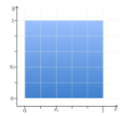
$$-\Delta = L \approx L_h$$

here: $\frac{du}{dx} = \frac{u(x_{i+1}) - u(x_i)}{\Delta x} + \mathcal{O}(\Delta x), \quad \Delta x = x_{i+1} - x_i.$

Grid:

$$x_i = i \cdot \Delta x$$
$$\Delta x = x_{i+1} - xi$$
$$i = 0 : N$$
$$N = \frac{1}{\Delta x}$$

y_j analogously



Discrete Operator:

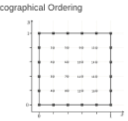
$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

Finite Difference Stencil:

Numbering:

Lexicographical Ordering



Matrix:

Linear System of Equations:

$$L_h u_h = f_h$$

where with $-\Delta u = f$

- L_h matrix of the last slide,
- u_h vector of all unknown grid values of u,
- f_h vector of all grid values of right hand side f.

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  - simple geometries and structured uniform meshes
- Finite Volumes

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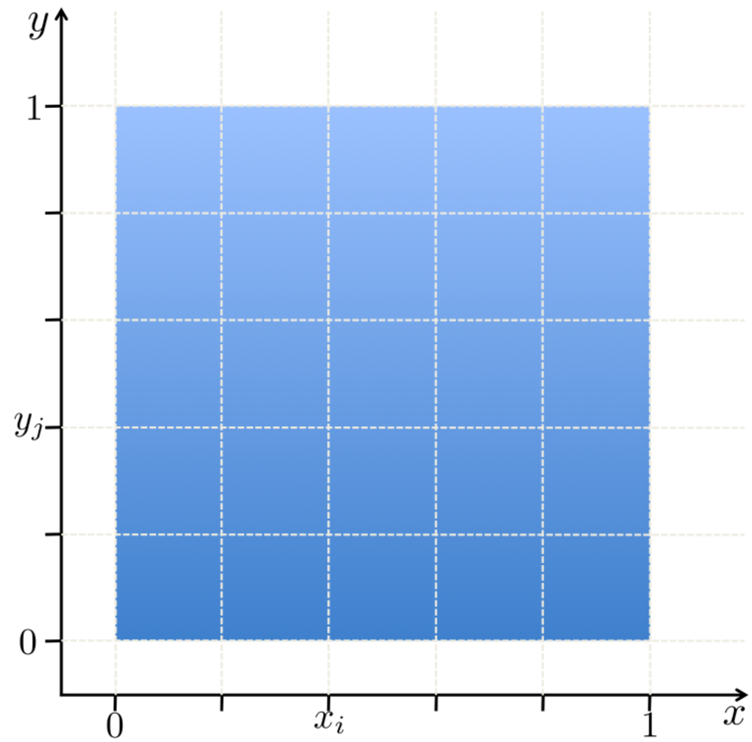

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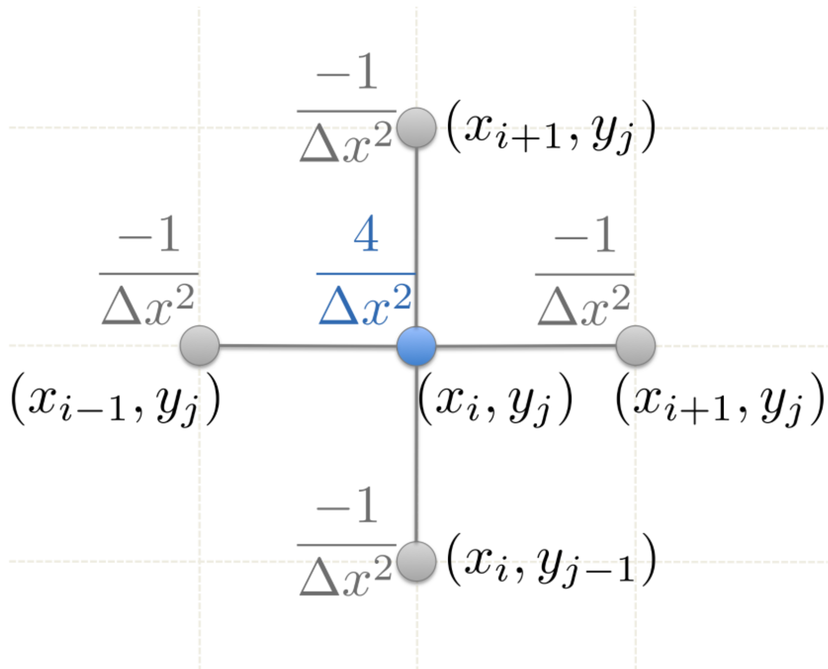
$$-\Delta u = \frac{4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

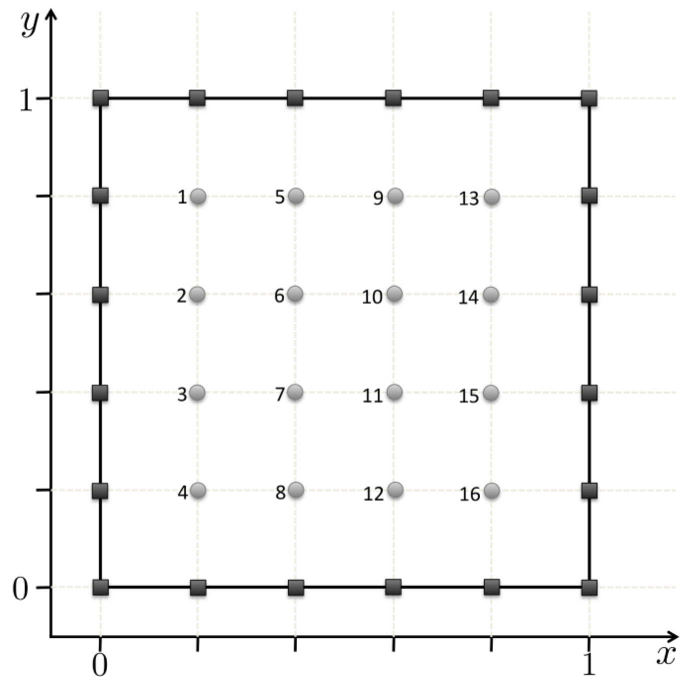
Finite Difference Stencil:

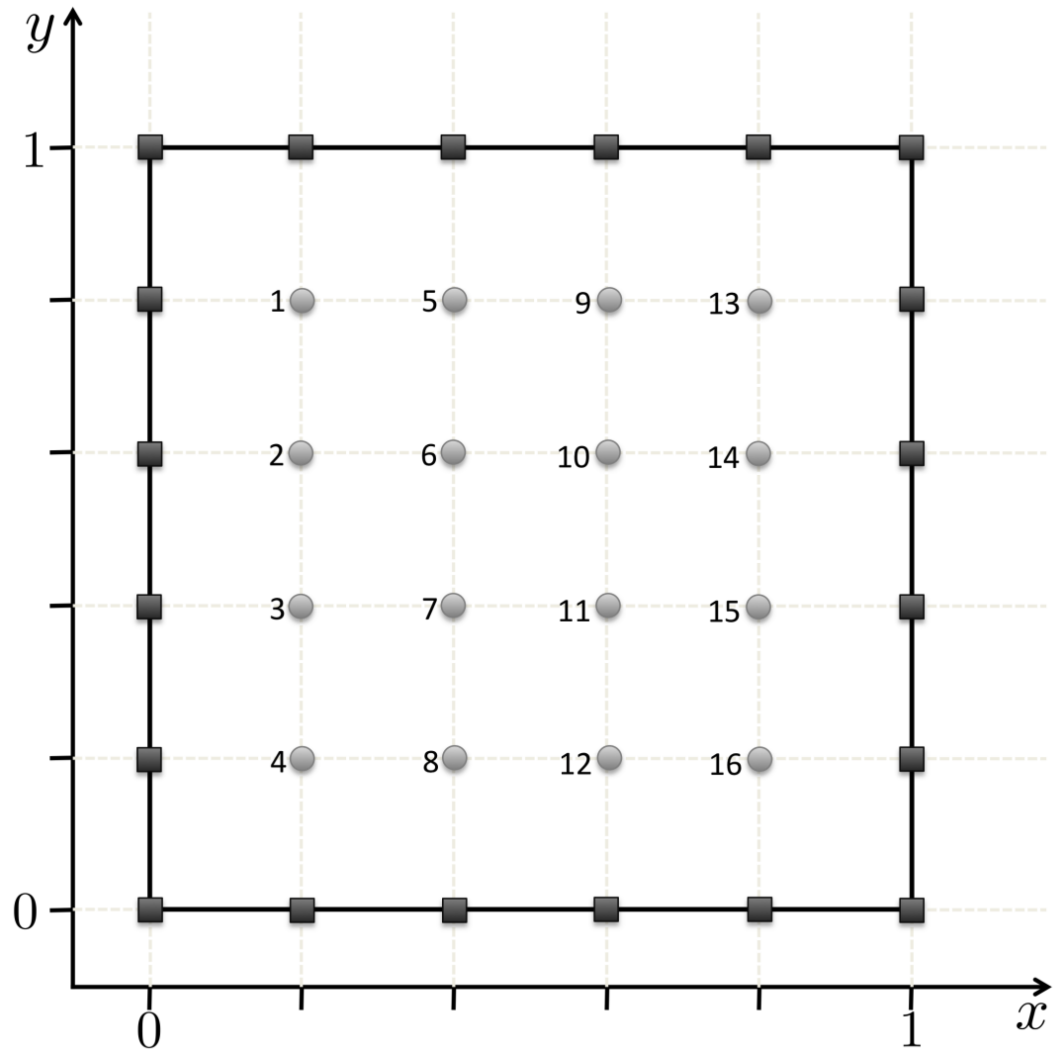


$$\rightarrow \frac{1}{\Delta x^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

Numbering:

Lexicographical Ordering




y
1
0
x
0
1
1
2
3
4
5
6
7
8
9
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14
15
16

Matrix:

$$\frac{1}{\Delta x^2}\begin{bmatrix}
4 & -1 & & & -1 & & & & & & & & & & & \\
-1 & 4 & -1 & & & -1 & & & & & & & & & & \\
& -1 & 4 & -1 & & & -1 & & & & & & & & & \\
& & -1 & 4 & & & & -1 & & & & & & & & \\
-1 & & & & 4 & -1 & & & -1 & & & & & & & \\
& -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
& & -1 & & & -1 & 4 & -1 & & & -1 & & & & & \\
& & & -1 & & & -1 & 4 & & & & -1 & & & & \\
& & & & -1 & & & & 4 & -1 & & & -1 & & & \\
& & & & & -1 & & & -1 & 4 & -1 & & & -1 & & \\
& & & & & & -1 & & & -1 & 4 & -1 & & & -1 & \\
& & & & & & & -1 & & & -1 & 4 & & & & -1 \\
& & & & & & & & -1 & & & & 4 & -1 & & \\
& & & & & & & & & -1 & & & -1 & 4 & -1 & \\
& & & & & & & & & & -1 & & & -1 & 4 & -1 \\
& & & & & & & & & & & -1 & & & -1 & 4
\end{bmatrix}$$

Linear System of Equations:

$$L_h u_h = f_h$$

where with $-\Delta u = f$

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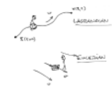
- u_h vector of all unknown grid values of u,
- f_h vector of all grid values of right hand side f.

Finite Volumes

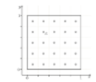
Idea:

Discretize the flux form



Cells (Grid):

Cover domain Ω by cells (volumes) $E_1, \dots, E_n$:



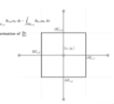
Integration:

$$\int_{E_i} \Delta u \, dx = \int_{E_i} f \, dx \quad \forall i = 1:N$$

| Gauß' theorem

$$-\int_{\partial E_i} \frac{\partial u}{\partial n} \, ds = \int_{E_i} f \, dx$$

Discretization



Simplified Notation:

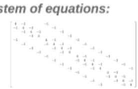
$u_{i,j} = u(x_i, y_j)$

$\bar{f}_{i,j} = \frac{1}{|E_{i,j}|} \int_{E_{i,j}} f \, dx$

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = \Delta x^2 \bar{f}_{i,j}.$$

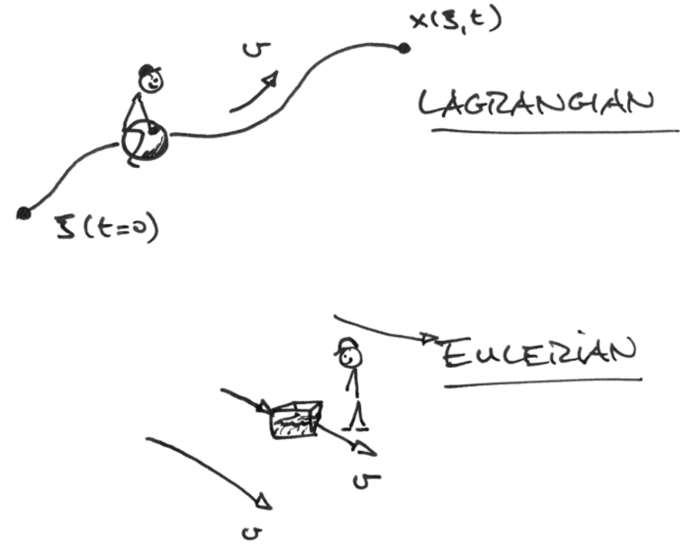
Linear system of equations:

$$L_h u_h = f_h$$



Idea:

Discretize the flux form




ξ(t=0)
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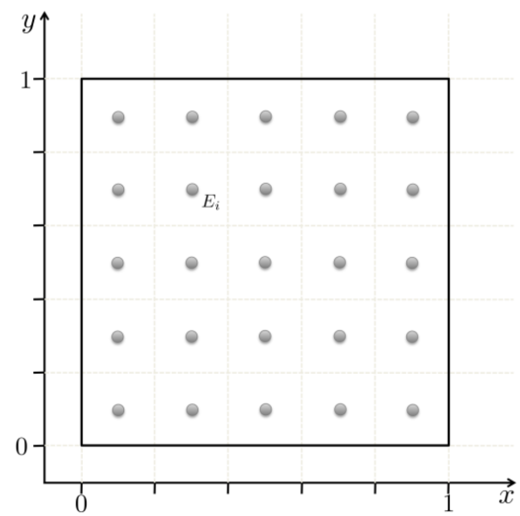
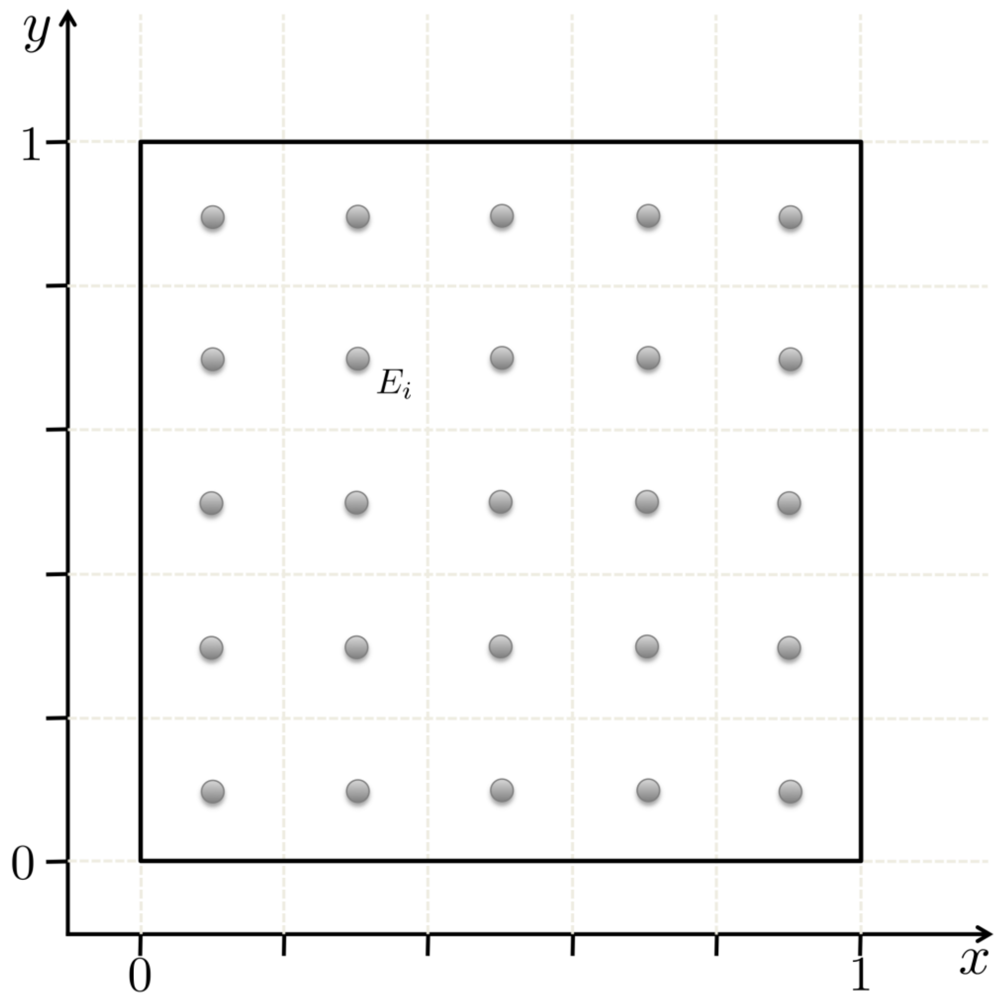
u

Cells (Grid):

Cover domain Ω by cells (volumes) $E_1, \ldots, E_n$:




y
1
0
E_i
0
1
x

Integration:

$$\int_{E_i} -\Delta u \; dx = \int_{E_i} f \; dx \quad \forall i = 1 : N$$

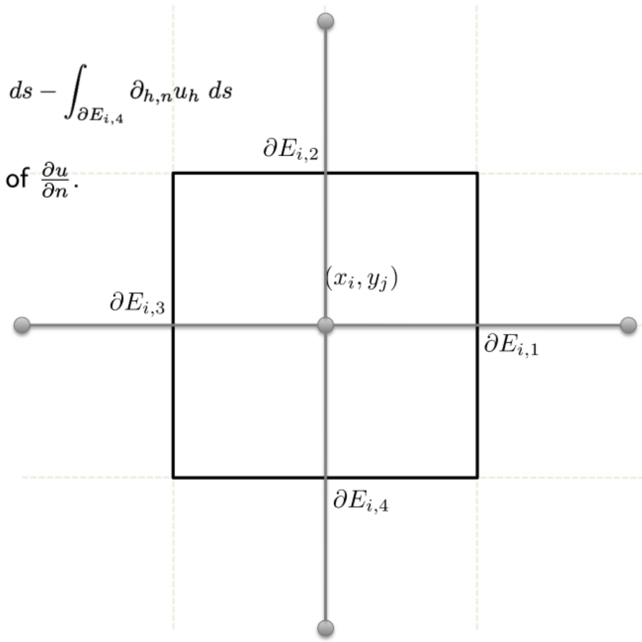
$$\| \qquad \text{Gauß}' \text{ theorem}$$

$$-\int_{\partial E_i} \frac{\partial u}{\partial n} \; ds = \int_{E_i} f \; dx$$

Discretization

$$\int_{E_i} f\, dx = -\int_{\partial E_i} \frac{\partial u}{\partial n}\, ds$$
$$\approx -\int_{\partial E_{i,1}} \partial_{h,n} u_h\, ds - \int_{\partial E_{i,2}} \partial_{h,n} u_h\, ds - \int_{\partial E_{i,3}} \partial_{h,n} u_h\, ds - \int_{\partial E_{i,4}} \partial_{h,n} u_h\, ds$$

$\partial_{h,n} u_h$: finite difference approximation of $\frac{\partial u}{\partial n}$.



$$\int_{\partial E_{i,1}} \partial_{h,n} u_h\, ds = \Delta x \cdot \left[\frac{u_h(x_{i+1}, y_j) - u_h(x_i, y_j)}{\Delta x}\right]$$

$$\int_{\partial E_{i,2}} \partial_{h,n} u_h\, ds = \Delta x \cdot \left[\frac{u_h(x_i, y_{j+1}) - u_h(x_i, y_j)}{\Delta x}\right]$$

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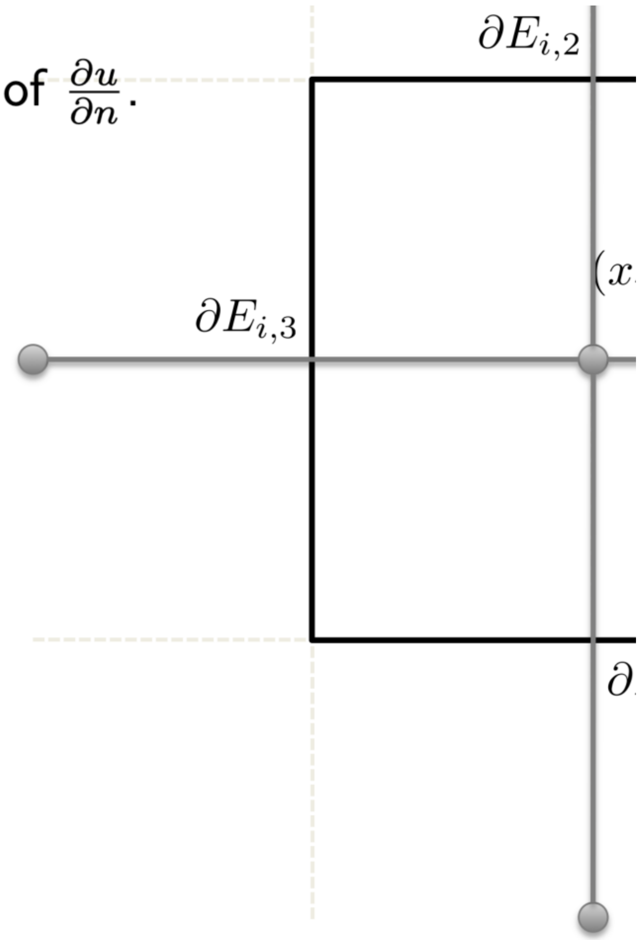

$$\int_{\partial E_{i,1}} \partial_{h,n} u_h \; ds = \Delta x \cdot \left[\frac{u_h(x_{i+1}, y_j) - u_h(x_i, y_j)}{\Delta x} \right]$$

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 & -1 & 4 & -1 & & & -1 & & & & & & & & & \\
 & & -1 & 4 & & & & -1 & & & & & & & & \\
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 & -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
 & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & \\
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 & & & & & & -1 & & & -1 & 4 & -1 & & & -1 & \\
 & & & & & & & -1 & & & -1 & 4 & & & & -1 \\
 & & & & & & & & -1 & & & & 4 & -1 & & \\
 & & & & & & & & & -1 & & & -1 & 4 & -1 & \\
 & & & & & & & & & & -1 & & & -1 & 4 & -1 \\
 & & & & & & & & & & & -1 & & & -1 & 4
\end{bmatrix}$$

where

- L_h matrix (right side),
- u_h vector of all unknown cell averages,
- f_h vector of all averaged right hand side values.

$$\int_{E_i} f \; dx \approx \Delta x^2 \cdot \overline{f(x_{E_i})}$$

Finite Elements

Idea:

"Discretize function space"

Finite Element

Variational Problem

Triangulation

Cover $\bar{\Omega}$ with set T_h of disjoint simplices:

- $\bar{\Omega} = \bigcup_{\tau \in T_h} \tau$
- If $\tau_1, \tau_2 \in T_h$ with $\tau_1 \neq \tau_2$, then $\mathring{\tau}_1 \cap \mathring{\tau}_2 = \emptyset$.
- If $\tau_1, \tau_2 \in T_h$ with $\tau_1 \neq \tau_2$, then

$$\tau_1 \cap \tau_2 = \begin{cases} \emptyset, \text{ or} \\ \text{common edge, or} \\ \text{common vertex.} \end{cases}$$

Replace Function (Spaces)

Instead of the problem

find $u \in V: \quad J(u) = \min_{v \in V} J(v)$

solve the discrete problem

find $u_h \in V_h: \quad J(u_h) = \min_{v_h \in V_h} J(v_h)$

Galerkin Method

Now, replace the variational problem $a(u_h, v_h) = f(v_h)$ by

$$u_i \cdot a(\varphi_i, \varphi_j) = f(\varphi_j), \quad \forall i, j = 1, \ldots, N,$$

since $u_h = \sum u_i \varphi_i$.

One obtains: $\quad L_h u_h = f_h$

where

- $(L_h)_{ij} = a(\varphi_i, \varphi_j)$ matrix,
- $u_h = [u_1, \ldots, u_N]^T$ vector of (unknown) coefficients,
- $(f_h)_i = f(\varphi_i)$ right hand side vector.

Funktion Space

- V: function space with $\dim V = \infty$;
- V_h piecewise polynomials, continuous in Ω, $\dim V_h = N < \infty$

Basis Function Representation

$$u_h(x) = \sum_{i=1:N} u_i \varphi_i(x)$$

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Variational Problem

Classical Problem

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$
$$\Rightarrow \int_\Omega -\Delta u \varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall\varphi$$
$$\Rightarrow \int_\Omega \nabla u \cdot \nabla\varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall\varphi$$

Variational Problem

$$a(u,v) = f(v) \quad \forall v$$

mit

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$
$$f(v) = \int_\Omega f v \, dx$$

Minimization Problem

find $u \in V: \quad J(u) = \min_{v \in V} J(v)$

with

$$J(v) = \frac{1}{2} a(v,v) - f(v)$$

Classical Problem

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$

$$u = 0 \quad \text{on } \partial\Omega$$

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Replace Function (Spaces)

Instead of the problem

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$$\text{find} \quad u_h \in V_h : \quad J(u_h) = \min_{v_h \in V_h} J(v_h)$$

Funktion Space

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Basis Function Representation

$$u_h(x) = \sum_{i=1:N} u_i \varphi_i(x)$$

$$V_h = \text{span}\{\varphi_1, \ldots, \varphi_N\}$$

$$v_h \in V_h \quad \Rightarrow \quad v_h = \sum_{i=1}^{N} v_i \varphi_i$$

where

- $v_i \in \mathbb{R}$ coefficients,
- $\varphi_i \in V_h$ basis polynomials.

Galerkin Method

Now, replace the variational problem $a(u_h, v_h) = f(v_h)$ by

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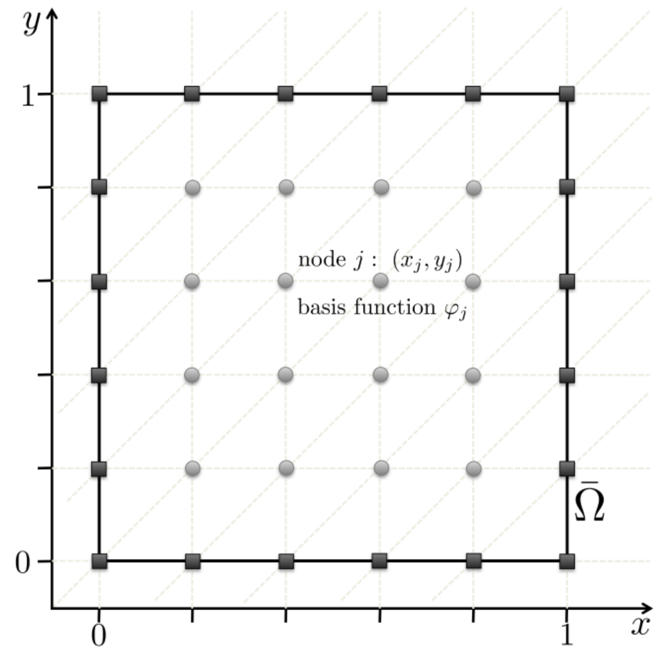
Triangulation

Cover $\overline{\Omega}$ with set T_h of disjoint simplices:

- $\overline{\Omega} = \bigcup_{\tau \in T_h} \tau$,
- If $\tau_1, \tau_2 \in T_h$ with $\tau_1 \neq \tau_2$, then $\mathring{\tau}_1 \cap \mathring{\tau}_2 = \emptyset$.
- If $\tau_1, \tau_2 \in T_h$ with $\tau_1 \neq \tau_2$, then

$$\overline{\tau_1} \cap \overline{\tau_2} = \begin{cases} \emptyset, \text{ or} \\ \text{common edge, or} \\ \text{common vertex.} \end{cases}$$



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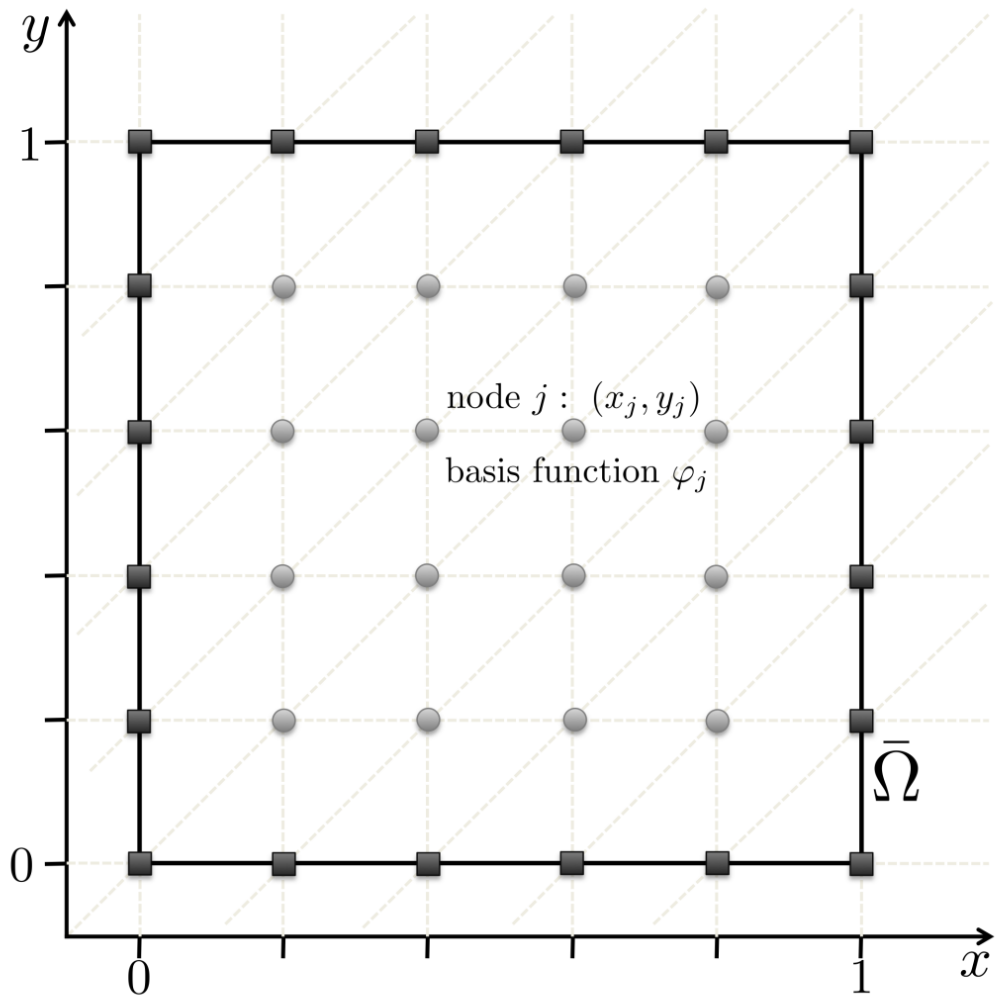
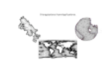
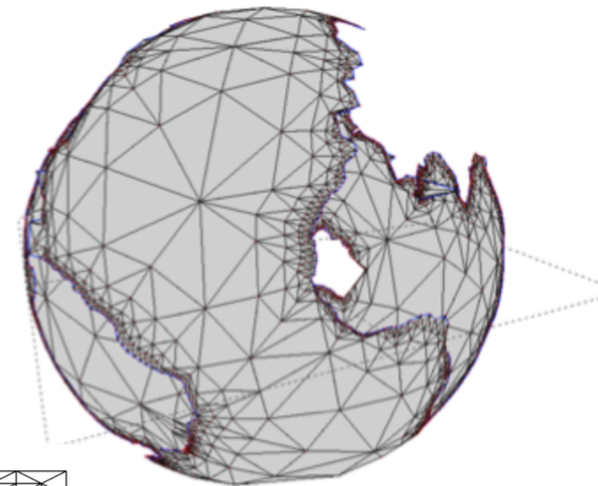
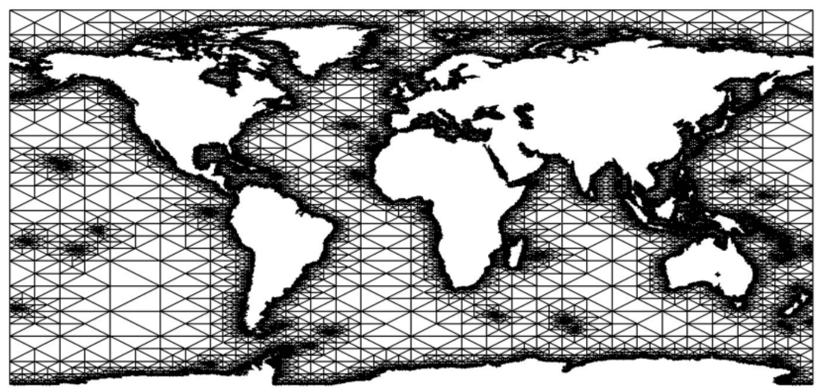
$\emptyset$, or
common edge, or
common vertex.





Triangulations from Applications:

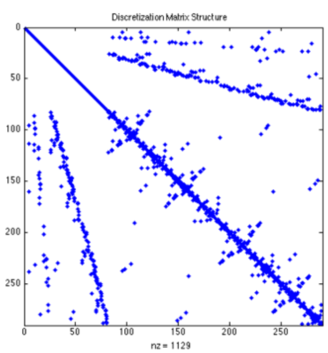


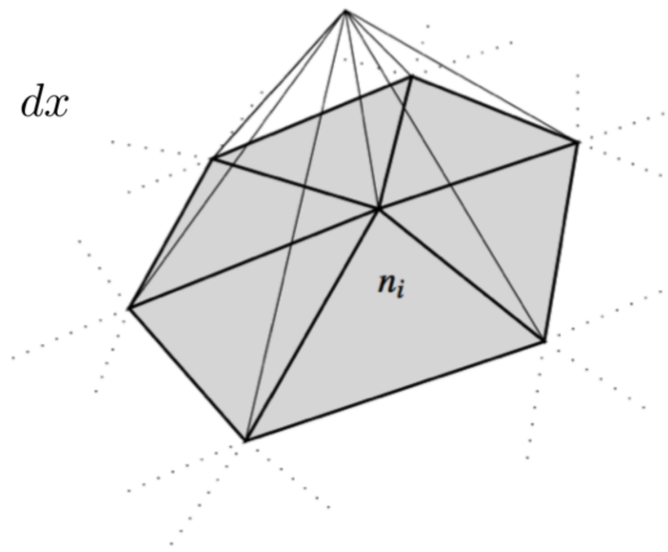


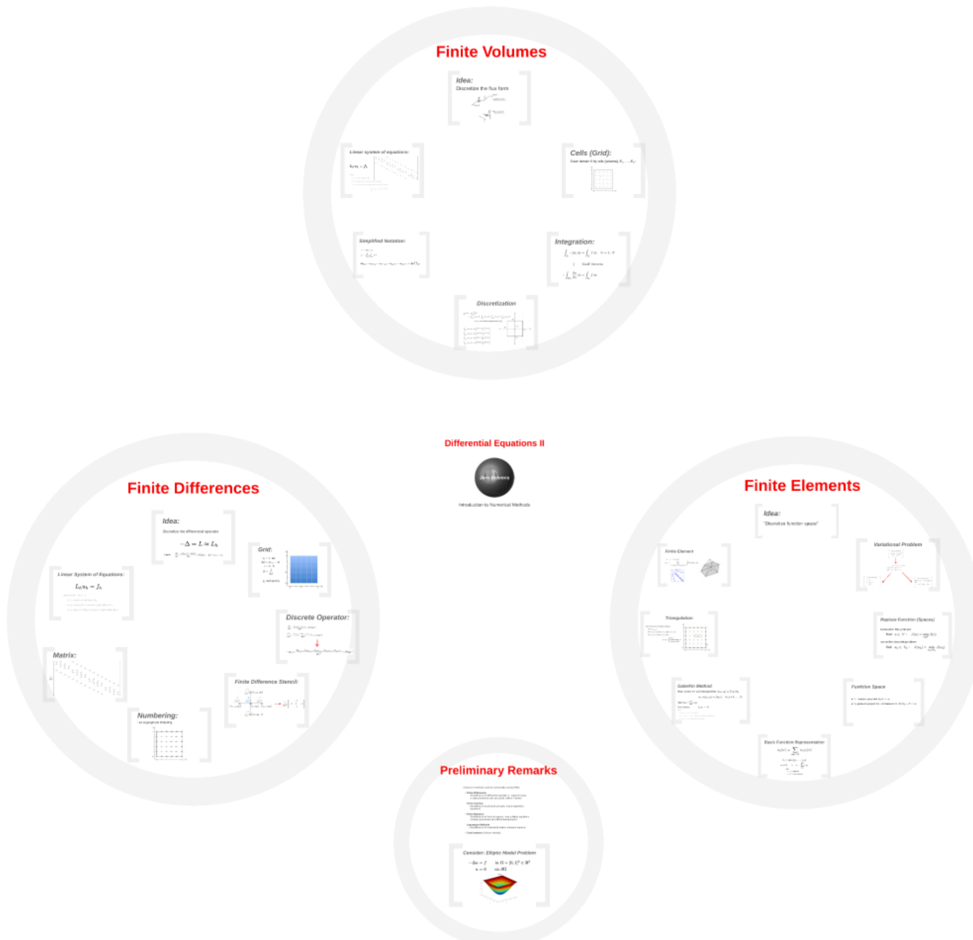
Finite Element

$$a_{ik} = a(\phi_i, \phi_k)$$

$$a_{ik} = \sum_{\tau \in (\mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j))} \int_\tau \nabla \phi_i \nabla \phi_j \, dx$$






Finite Volumes
Differential Equations II
Finite Differences
Finite Elements
Preliminary Remarks