Differential Equations II

Wrap Up

1. Introduction

Physical Principles of PDEs

Conservation Principle

Idea: Derive PDE from physical law (conservation)!

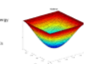
Mass M in control volume V

$$M = \int_V \rho(x)\ dx$$

Variational Principle

Idea: Derive PDE from physical law (Minimization)!

Minimization problem: minimize bending energy



Continuity Equation

General Definition

Definition: (Partial Differential Equation)
An equation resp. a system of equations of the form

$$\mathbf{F}\left(\mathbf{x}, \mathbf{u}(\mathbf{x}), \frac{\partial \mathbf{u}}{\partial x_1}, \ldots, \frac{\partial \mathbf{u}}{\partial x_n}, \ldots, \frac{\partial^p \mathbf{u}}{\partial^p x_1}, \frac{\partial^p \mathbf{u}}{\partial^{p-1} x_1 \partial x_2}, \ldots, \frac{\partial^p \mathbf{u}}{\partial^p x_n}\right) = \mathbf{0}$$

for an unknown function $\mathbf{u} : D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$, is called system of partial differential equations (PDE) for the m functions $u_1(\mathbf{x}), \ldots, u_m(\mathbf{x})$.

If one of the partial derivatives occurs explicitly and is of p^{th} order ($\frac{\partial^p \mathbf{u}}{\partial^{p_1} x_1 \cdots \partial^{p_n} x_n}$), then we call the PDE of order p.

Remark: In applications we see typically (systems of) PDE of first and second order.

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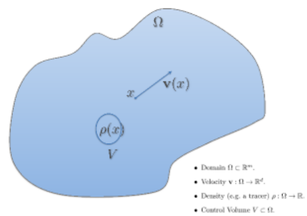


Variational Principle

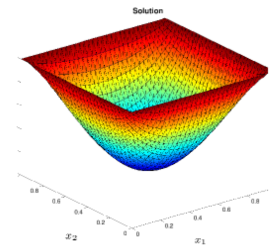
Idea: Derive PDE from physical law (Minimization)!

Membrane = Minimal surface

Minimization problem: minimize bending energy

$$J = \int_\Omega \sqrt{1 + \partial_{x_1} u^2 + \partial_{x_2} u^2}\ dx_1 dx_2 \stackrel{!}{=} \min$$

$$u|_{\partial\Omega} = \phi$$



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(systems of) PDE of first and second order.

Continuity Equation

Continuity Equation:

- Let $\rho(\mathbf{x},t)$ be mass density of a physical constituent (e.g. fluid density).
- Assume a conservation principle of the form

$$\frac{d}{dt}\int_{D_t}\rho(\mathbf{x},t)\ d\mathbf{x}=0.$$

- According to Reynold's transport theorem it holds:

$$\int_{D_t}\left[\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\mathbf{v})\right](\mathbf{x},t)\ d\mathbf{x}=0.$$

- Since $D_t\subset\mathbb{R}^n$ arbitrary subset, the PDE (continuity equation) holds:

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t)+\nabla\cdot(\rho\mathbf{v})(\mathbf{x},t)=0.$$

Flux Function:

- Write the continuity eq. by means of a flux function $\mathbf{q}(\mathbf{x},t)$:

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t)+\nabla\cdot\mathbf{q}(\mathbf{x},t)=0.$$

- Avoid *two* unknowns ρ and $\mathbf{q}$ in *one* equation by

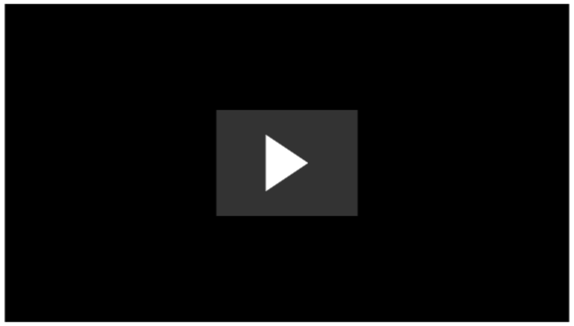
$$\mathbf{q}(\mathbf{x},t)=\mathbf{q}(\rho(\mathbf{x},t),\nabla\rho(\mathbf{x},t),\ldots).$$

- Example: the flux $\mathbf{q}$ is proportional to the density ρ, i.e.

$$\mathbf{q}(\mathbf{x},t)=\mathbf{a}\cdot\rho(\mathbf{x},t),\quad \mathbf{a}\in\mathbb{R}^n.$$

- Then we obtain the (transport equation):

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t)+\mathbf{a}\cdot\nabla\rho(\mathbf{x},t)=0.$$



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2. Method of Characteristics

Definition: The autonomous system of ODEs

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t))$$

Is called **Characteristic System of Differential Equations** corresponding to a homogeneous linear PDE

$$\sum_{i=1}^{n} a_i(\mathbf{x}) u_{x_i} = 0, \quad \mathbf{x} \in \mathbb{R}^n.$$

Definition (Cauchy Problem):
The initial value problem defined on the whole $\mathbb{R}^n$

$$\begin{cases} u_t + \sum_{i=1}^{n} a_i(\mathbf{x}, t, u) u_{x_i} = b(\mathbf{x}, t, u) & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = u_0 & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

is called Cauchy Problem.

Example: Consider the PDE in three variables

$$x u_x + y u_y + (x^2 + y^2) u_z = 0.$$

The Characteristic system of ODEs reads:

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= y \\ \dot{z} &= x^2 + y^2 \end{aligned}$$

The general solution is

$$\begin{aligned} x(t) &= c_1 e^t \\ y(t) &= c_2 e^t \\ z(t) &= \frac{1}{2}(c_1^2 + c_2^2) e^{2t} + c_3 \end{aligned}$$

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3. Conservationlaws

Example: (Burgers' Equation)
Burgers' Gleichung is given by the flux function $f(u) = \frac{u^2}{2}$, resp. by the Cauchy problem

$$\begin{aligned} u_t + uu_x &= 0 \quad \text{in } \mathbb{R} \times]0, \infty[\\ u &= u_0 \quad \text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

- The solution is given by $x(t) = x_0 + tu_0(x_0)$.
- If u_0 is given by
$$u_0(x) = \begin{cases} 1 & : \ x \leq 0 \\ 1 - x & : \ 0 < x < 1 \\ 0 & : \ x \geq 1 \end{cases}$$
then $x(t)$ develops a singularity for $t \to 1$.
- A classical solution of Burgers' equation exists only locally for $0 \leq t < 1$.
- The local solution for $t \in [0, 1[$ is:
$$u(x,t) = \begin{cases} 1 & : \ x < 1 \\ \frac{(1-x)}{(1-t)} & : \ 0 \leq t \leq x \leq 1 \\ 0 & : \ x > 1 \end{cases}$$

Definition: (weak solution)
A function $u \in L^\infty(\mathbb{R} \times [0, \infty[)$ is called integral solution or weak solution of the conservation law $u_t + f(u)_x = 0$, if for all test functions v:

$$\int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x) \, dx dt + \int_{-\infty}^\infty u_0(x) v(x,0) \, dx = 0.$$

Definition: (shock wave solution)
A shock wave solution u is a weak solution of the conservation law

$$u_t + f(u)_x = 0$$

if a shock front $x = s(t)$, $s \in C^1$ exists, such that u is a classical solution for each $x < s(t)$ and $x > s(t)$ and u has a jump at $x = s(t)$ with height

$$[u](t) = u(s(t)^+, t) - u(s(t)^-, t) = u_r - u_l.$$

$\dot{s}(t)$ is called shock speed.

Proposition: (Rankine-Hugoniot condition)
If $x = s(t)$ is a shock front of a shock wave solution of $u_t + f(u)_x = 0$, then for the shock speed $\dot{s}(t)$ the Rankine-Hugoniot condition holds:

$$\dot{s} = \frac{[f]}{[u]} = \frac{f(u(s(t)^-, t)) - f(u(s(t)^+, t))}{u(s(t)^-, t) - u(s(t)^+, t)} = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

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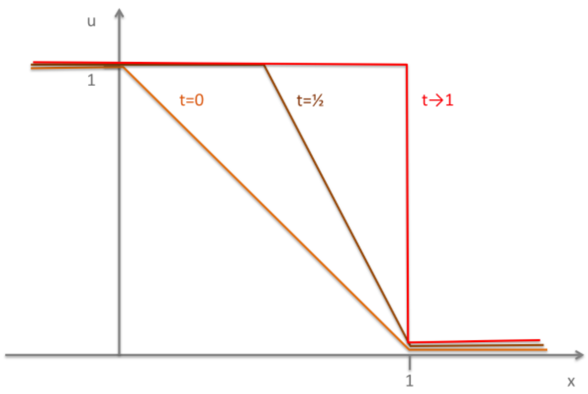

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4. Entropy Condition

Proposition: (Rarefaction Wave)
Let the Riemann problem with Burgers' equation $u_t + uu_x = 0$ in $\mathbb{R}\times]0,\infty[$ and $u(x,t=0) = x_0$ be given. Let

$$u_0(x) = \begin{cases} u_l & : \quad x \leq 0 \\ u_r & : \quad x > 0 \end{cases} \quad \text{with } u_l < u_r.$$

Then the rarefaction wave is given by

$$u(x,t) = \begin{cases} u_l & : \quad x < f'(u_l)t \\ g(\frac{x}{t}) & : \quad f'(u_l)t \leq x \leq f'(u_r)t \\ u_r & : \quad x > f'(u_r)t \end{cases}$$

an integral solution of the Riemann problem.

Definition: (Entropy Condition)
An integral solution is called entropy solution, if the solution fulfills the entropy condition or Lax-Oleinik condition:
There exists $C > 0$, such that for all $x, z \in \mathbb{R}$, $t > 0$ with $z > 0$ it holds:

$$u(t, x+z) - u(t,x) < \frac{C}{t} z.$$

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5. PDEs of Second Order

Definition: (PDE of 2nd Order)
A linear partial differential equation of 2nd order in n variables is defined by

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + fu = g.$$

Here the terms a_{ij}, b_i, f, and g are functions of $\mathbf{x} = (x_1, \ldots, x_n)^\top$.

Definition: (Well-Posed Problem)
A correctly posed problem (or well-posed problem) consists of

- a *partial differential equation*, defined on a domain, and
- a set of *initial and/or boundary conditions*,

such that the following properties are fulfilled:

1. Existence: There exists at least one solution, that fulfills all above conditions;
2. Uniqueness: The solution is unique;
3. Stability: The solution depends cont. on the initial/boundary conditions

Definition: (Classification of Partial Differential Equations of 2nd Order)
Let the PDE of 2nd order ($A = (a_{ij})_{i,j=1,\ldots,n}$ constant and symmetric)

$$(\nabla^\top A \nabla)u + (\mathbf{b}^\top \nabla)u + fu = g.$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of matrix A.

1. If $\lambda_i \neq 0$ for all $i = 1, \ldots, n$ and if all λ_i have equal sign, the equation is called elliptic.
2. If $\lambda_i \neq 0$ for all $i = 1, \ldots, n$ and if one eigenvalue has different sign to all other $n-1$ eigenvalues, the equation is called hyperbolic.
3. If $\lambda_k = 0$ for at least one $k \in \{1, \ldots, n\}$, the equation is called parabolic.

1. The elliptic Laplace equation
$$\Delta u = 0.$$
2. The hyperbolic wave equation
$$u_{tt} - \Delta u = 0.$$
3. The parabolic heat equation
$$u_t = \Delta u.$$

ation)

$(n = 2)$
$(n \geq 3)$

Definition: (PDE of 2^{nd} Order)
A linear partial differential equation of 2^{nd} order in n variables is defined by

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6. Laplace's Equation

Definition: (Laplace's and Poisson's Equation)
Let $u \in C^2(\mathbb{R}^n)$ be a twice cont. differentiable function, $\mathbf{x} \in D \subset \mathbb{R}^n$ open, $u = u(\mathbf{x})$. Then Laplace's equation is given by

$$\Delta u = 0.$$

Poisson's equation is defined as

$$-\Delta u = f$$

with a given right hand side $f = f(\mathbf{x})$.

Definition: (Green's Function)
Let $U \subset \mathbb{R}^n$ be open and $\Phi^x(\mathbf{y})$ the solution of Dirichlet's Problem

$$\begin{aligned} \Delta \Phi^x &= 0 \quad \text{in } U \\ \Phi^x &= \Phi(\mathbf{y} - \mathbf{x}) \quad \text{on } \partial U. \end{aligned}$$

Then Green's function G on U is defined by

$$G(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{y} - \mathbf{x}) - \Phi^x(\mathbf{y}) \quad \mathbf{x}, \mathbf{y} \in U, \mathbf{x} \neq \mathbf{y}.$$

Definition: (Fundamental Solution of Laplace's Equation)
The function $\Phi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$, given by

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\pi} \log \|x\| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \|x\|^{2-n} & (n \geq 3) \end{cases}$$

is called fundamental solution of Laplace's equation.

Proposition: (Mean Value Property of Harmonic Functions)
Let $U \subset \mathbb{R}^n$ be an open set. If $u \in C^2(U)$ is harmonic, then for each ball $B(\mathbf{x}, r) \subset U$

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} u \, dS = \fint_{B(\mathbf{x},r)} u \, d\mathbf{y}.$$

Proposition: (Unique Solvability of Boundary Value Problem)
Let $g \in C(\partial U)$ and $f \in C(U)$. Then there is at most one solution $u \in C^2(U) \cap C(\overline{U})$ of the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= g \quad \text{on } \partial U. \end{aligned}$$

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Proposition: (Representation of Solution of Poisson's Equation)
A solution to Poisson's equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^n$$

is given by

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}.$$

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Then Green's function G on U is defined by

$$G(\mathbf{x},\mathbf{y}) := \Phi(\mathbf{y}-\mathbf{x}) - \Phi^x(\mathbf{y}) \quad \mathbf{x},\mathbf{y} \in U, \mathbf{x} \neq \mathbf{y}.$$

Proposition: (Solution of Dirichlet Problem of Poisson's Equation)
Let $u \in C^2(\overline{U})$ be a solution of the Dirichlet problem of Poisson's equation. Then u can be represented as

$$u(\mathbf{x}) = \int_{\partial U} g(\mathbf{y}) \frac{\partial G}{\partial \mathbf{n}}(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}) + \int_U f(\mathbf{y}) G(\mathbf{x},\mathbf{y})\, d\mathbf{y} \quad (\mathbf{x} \in U).$$

f and g are the right hand side and boundary condition of the Dirichlet problem.

7. Green's Function

Definition: (Poisson Kernel)
The function

$$K(\mathbf{x},\mathbf{y}) := \frac{2x_n}{n\alpha(n)} \frac{1}{|\mathbf{x}-\mathbf{y}|^n},$$

where $\mathbf{x} \in \mathbb{R}^n_+$, $\mathbf{y} \in \partial\mathbb{R}^n_+$ is called Poisson Kernel of $\mathbb{R}^n_+$.

Proposition: (Dirichlet Problem for Laplace's Equation)
Let the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+ \\ u = g & \text{on} \quad \partial\mathbb{R}^n_+ = \{\mathbf{x} = (x_1,\ldots,x_n)^\top : x_n = 0\} \end{cases}$$

be given. Then the solution is given by Poisson's integral form

$$u(\mathbf{x}) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^n}\, d\mathbf{y}.$$

Definition: (Poisson Kernel)
The function

$$K(\mathbf{x}, \mathbf{y}) := \frac{2x_n}{n\alpha(n)} \frac{1}{|\mathbf{x} - \mathbf{y}|^n},$$

where $\mathbf{x} \in \mathbb{R}^n_+$, $\mathbf{y} \in \partial\mathbb{R}^n_+$ is called Poisson Kernel of $\mathbb{R}^n_+$.

Proposition: (Dirichlet Problem for Laplace's Equation)
Let the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+ \\ u = g & \text{on} \quad \partial\mathbb{R}^n_+ = \{\mathbf{x} = (x_1, \ldots, x_n)^\top : x_n = 0\} \end{cases}$$

be given. Then the solution is given by Poisson's integral form

$$u(\mathbf{x}) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} \, d\mathbf{y}.$$

8. Heat Equation

Definition: (Fundamental Solution to Heat Equation) The function

$$\Phi(\mathbf{x},t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{|x|^2}{4t}} & (\mathbf{x} \in \mathbb{R}^n, t > 0) \\ 0 & (\mathbf{x} \in \mathbb{R}^n, t < 0) \end{cases}$$

is called fundamental solution of the heat equations.

Remark: (Solution to the Cauchy Problem)
By means of $\Phi(\mathbf{x},t)$ the solution to the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times]0,\infty[\\ u = g & \text{on} \quad \mathbb{R}^n \times \{0\} \end{cases}$$

can be represented by a convolution integral:

$$u(\mathbf{x},t) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t) g(\mathbf{y}) \, d\mathbf{y}$$

Proposition: (Mean Value Property of Heat Equation)
If $u \in C_1^2(U_t)$ is a solution of the heat equation, then

$$u(\mathbf{x},t) = \frac{1}{4r^n} \int_{E(\mathbf{x},t;r)} \frac{|\mathbf{x}-\mathbf{y}|^2}{(t-s)^2} u(\mathbf{y},s) \, d\mathbf{y} ds$$

for each set $E(\mathbf{x},t;r) \subset U_t$.

Proposition: (Unique Solution of Heat Equation)
The initial value problem

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } U_t \\ u &= g \quad \text{auf } \Gamma_T \end{cases}$$

on the bounded domain U_T with continuous functions f and g has at most one solution $u \in C_1^2(U_T) \cap C(\overline{U_T})$.

The inhomogeneous initial value problem with inhomogeneous initial conditions

$$\begin{cases} u_t - \Delta u &= f \quad \text{in } \mathbb{R}^n \times]0,\infty[\\ u(\mathbf{x},0) &= g(\mathbf{x}) \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

has the solution

$$u(\mathbf{x},t) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t) g(\mathbf{y}) \, d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t-s) f(\mathbf{y},s) \, d\mathbf{y} ds.$$

Definition: (Fundamental Solution to Heat Equation) The function

$$\Phi(\mathbf{x},t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{|x|^2}{4t}} & (\mathbf{x} \in \mathbb{R}^n, t > 0) \\ 0 & (\mathbf{x} \in \mathbb{R}^n, t < 0) \end{cases}$$

is called fundamental solution of the heat equations.

Remark: (Solution to the Cauchy Problem)
By means of $\Phi(\mathbf{x},t)$ the solution to the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times]0,\infty[\\ u = g & \text{on} \quad \mathbb{R}^n \times \{0\} \end{cases}$$

can be represented by a convolution integral:

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has the solution

$$u(\mathbf{x},t) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t)g(\mathbf{y})\ d\mathbf{y} + \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t-s)f(\mathbf{y},s)\ d\mathbf{y}ds.$$

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on the bounded domain U_T with continuous functions f and g
has at most one solution $u \in C_1^2(U_T) \cap C(\overline{U_T})$.

9. Wave Equation

Proposition: (Formula of d'Alembert)
A solution of the one-dimensional initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times [0, \infty[, \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

with g, h initial conditions, is given by the formula of d'Alembert:

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy.$$

Conclusion: (Reflection of half space $\mathbb{R}_+$)
A solution of the *initial value problem*

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times]0, \infty[\\ u = g, u_t = h & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 & \text{on } \{x = 0\} \times]0, \infty[\end{cases}$$

is given by

$$u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy & \text{for } x \geq t \geq 0 \\ \frac{1}{2}[g(x+t) - g(-x+t)] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\, dy & \text{for } 0 \leq x \leq t \end{cases}$$

Remark: (Poisson's Formula for $n = 2$)
The solution of the initial value problem of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^2 \times]0, \infty[\\ u = g,\ u_t = h & \text{on } \mathbb{R}^2 \times \{t = 0\} \end{cases}$$

for $x \in \mathbb{R}^2$, $t > 0$ is given by (Poisson's formula):

$$u(x,t) = \frac{1}{2} ⨍_{\partial B_t(x)} \frac{t g(y) + t^2 h(y) + t Dg(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{\frac{1}{2}}}\, dy$$

Remark: (Mean/Average over Sphere) For $x \in \mathbb{R}^n$, $t > 0$ and $r > 0$ define the Average of $u(x,t)$ over the Sphere $\partial B_r(x)$ (or $\partial B(x,r)$)

$$U(x; r, t) := ⨍_{\partial B_r(x)} u(y,t)\, dS(y).$$

Furthermore, let

$$G(x; r) := ⨍_{\partial B_r(x)} g(y)\, dS(y)$$
$$H(x, r) := ⨍_{\partial B_r(x)} h(y)\, dS(y)$$

Remark: (Kirchhoff's Formula for $n = 3$)
The solution to the initial value problem of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times]0, \infty[\\ u = g,\ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

for $x \in \mathbb{R}^3$, $t > 0$ is given by (Kirchhoff's Formula):

$$u(x,t) = ⨍_{\partial B_t(x)} (t h(y) + g(y) + Dg(y) \cdot (y - x))\, dS(y)$$

Proposition: (Euler-Poisson-Darboux Equation)
let $x \in \mathbb{R}^n$ be fixed and u a solution of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times]0, \infty[\\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Then $U(x; r, t)$ solves the Euler-Poisson-Darboux equation

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times]0, \infty[\\ U = G,\ U_t = H & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$$

Proposition: (Formula of d'Alembert)
A solution of the one-dimensional initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times [0, \infty[, \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

with g, h initial conditions, is given by the formula of d'Alembert:

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy.$$

Conclusion: (Reflection of half space $\mathbb{R}_+$)
A solution of the *initial value problem*

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times]0,\infty[\\ u = g, u_t = h & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 & \text{on } \{x = 0\} \times]0,\infty[\end{cases}$$

is given by

$$u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy & \text{for } x \geq t \geq 0 \\ \frac{1}{2}[g(x+t) - g(-x+t)] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\, dy & \text{for } 0 \leq x \leq t \end{cases}$$

Remark: (Mean/Average over Sphere) For $x \in \mathbb{R}^n$, $t > 0$ and $r > 0$ define the Average of $u(x,t)$ over the Sphere $\partial B_r(x)$ (or $\partial B(x,r)$)

$$U(x;r,t) := \fint_{\partial B_r(x)} u(y,t)\, dS(y).$$

Furthermore, let

$$\begin{aligned} G(x;r) &:= \fint_{\partial B_r(x)} g(y)\, dS(y) \\ H(x,r) &:= \fint_{\partial B_r(x)} h(y)\, dS(y) \end{aligned}$$

Proposition: (Euler-Poisson-Darboux Equation)
let $x \in \mathbb{R}^n$ be fixed and u a solution of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times]0, \infty[\\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Then $U(x; r, t)$ solves the Euler-Poisson-Darboux equation

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Remark: (Kirchhoff's Formula for $n = 3$)
The solution to the initial value problem of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times]0, \infty[\\ u = g, \ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

for $x \in \mathbb{R}^3$, $t > 0$ is given by (Kirchhoff's Formula):

$$u(x,t) = \fint_{\partial B_t(x)} (th(y) + g(y) + Dg(y) \cdot (y - x)) \, dS(y)$$

Remark: (Poisson's Formula for $n = 2$)
The solution of the initial value problem of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^2 \times]0, \infty[\\ u = g, \ u_t = h & \text{on } \mathbb{R}^2 \times \{t = 0\} \end{cases}$$

for $x \in \mathbb{R}^2$, $t > 0$ is given by (Poisson's formula):

$$u(x,t) = \frac{1}{2} \fint_{\partial B_t(x)} \frac{tg(y) + t^2 h(y) + tDg(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} \, dy$$

10. Fourier Method

Remark: (General Approximate Solution of 1D Poisson's Equation)

- Let the one-dimensional boundary value problem be given:

$$\begin{cases} -T\frac{d^2u}{dx^2} &= f(x), \quad 0 < x < l, \\ u(0) = u(l) &= 0. \end{cases}$$

- Approximate the right hand side $f(x)$ by a finite Fourier series $f_N(x)$:

$$f_N(x) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{l}\right).$$

- The Fourier coefficients are $(n = 1, \ldots, N)$

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\, dx.$$

- Then an approximate solution of the boundary value problem is given by:

$$u_N(x) = \sum_{n=1}^{N} \frac{l^2 c_n}{Tn^2\pi^2}\sin\left(\frac{n\pi x}{l}\right).$$

Die Lösung des Anfangsrandwertproblems

$$\begin{cases} u_{tt} - u_{xx} = 0 & : 0 < x < l, 0 < t \leq T \\ u(x,0) = g(x) & : 0 \leq x \leq l \\ u_t(x,0) = h(x) & : 0 \leq x \leq l \\ u(0,t) = u(l,t) = 0 & : 0 \leq t \leq T \end{cases}$$

ist gegeben durch

$$u(x,t) = \sum_{n=1}^{\infty}\left\{b_n \cos\left(\frac{n\pi}{l}t\right) + \frac{d_n l}{n\pi}\sin\left(\frac{n\pi}{l}t\right)\right\}\sin\left(\frac{n\pi x}{l}\right)$$

Dabei sind b_n die Fourier–Koeffizienten der Entwicklung der vorgegeben Anfangsbedingung $u(x,0) = g(x)$ und d_n die entsprechenden Koeffizienten von $u_t(x,0) = h(x)$.

Remember: (Heat Equation) Consider the initial boundary value problem of the heat equation:

$$\begin{cases} u_t - u_{xx} = f(x,t) & : 0 < x < l, 0 < t \leq T, \\ u(x,0) = g(x) & : 0 \leq x \leq l, \\ u(0,t) = u(l,t) = 0 & : 0 \leq t \leq T. \end{cases}$$

We look for a solution in form of a Fourier series:

$$u_N(x) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{n\pi x}{l}\right).$$

Periodic Boundary Conditions

Let the initial boundary value problem be given on **interval** $[-l,l]$:

$$\begin{cases} u_t - u_{xx} = f(x,t) & : -l < x < l, 0 < t \leq T, \\ u(x,0) = g(x) & : -l \leq x \leq l, \\ u(-l,t) = u(l,t) & : 0 \leq t \leq T, \\ u_x(-l,t) = u_x(l,t) & : 0 \leq t \leq T. \end{cases}$$

Periodic functions on $[-l,l]$ are

$$\psi(x) = \frac{1}{2}, \quad \psi(x) = \cos\left(\frac{n\pi x}{l}\right), \quad \psi(x) = \sin\left(\frac{n\pi x}{l}\right)$$

satisfy the given Neumann boundary conditions.
A **solution approach** using Fourier series thus reads

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty}\left(a_n(t)\cos\left(\frac{n\pi x}{l}\right) + b_n(t)\sin\left(\frac{n\pi x}{l}\right)\right).$$

Remark: (General Approximate Solution of 1D Poisson's Equation)

- Let the one-dimensional boundary value problem be given:

$$\begin{cases} -T\frac{d^2u}{dx^2} = f(x), & 0 < x < l, \\ u(0) = u(l) = 0. \end{cases}$$

- Approximate the right hand side $f(x)$ by a finite Fourier series $f_N(x)$:

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- The Fourier coefficients are $(n = 1, \ldots, N)$

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\, dx.$$

- Then an approximate solution of the boundary value problem is given by:

$$u_N(x) = \sum_{n=1}^{N} \frac{l^2 c_n}{T n^2 \pi^2}\sin\left(\frac{n\pi x}{l}\right).$$

Remember: (Heat Equation) Consider the initial boundary value problem of the heat equation:

$$\begin{cases} u_t - u_{xx} = f(x,t) & : \; 0 < x < l, \; 0 < t \leq T, \\ u(x,0) = g(x) & : \; 0 \leq x \leq l, \\ u(0,t) = u(l,t) = 0 & : \; 0 \leq t \leq T. \end{cases}$$

We look for a solution in form of a Fourier series:

$$u_N(x) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right).$$

Periodic Boundary Conditions

Let the initial boundary value problem be given on **interval** $[-l, l]$:

$$\begin{cases} u_t - u_{xx} & = & f(x,t) & : & -l < x < l,\ 0 < t \leq T, \\ u(x,0) & = & g(x) & : & -l \leq x \leq l, \\ u(-l,t) & = & u(l,t) & : & 0 \leq t \leq T, \\ u_x(-l,t) & = & u_x(l,t) & : & 0 \leq t \leq T. \end{cases}$$

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Die Lösung des Anfangsrandwertproblems

$$\begin{cases} u_{tt} - u_{xx} &= 0 &: 0 < x < l,\, 0 < t \le T \\ u(x,0) &= g(x) &: 0 \le x \le l \\ u_t(x,0) &= h(x) &: 0 \le x \le l \\ u(0,t) &= u(l,t) = 0 &: 0 \le t \le T \end{cases}$$

ist gegeben durch

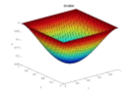
$$u(x,t) = \sum_{n=1}^{\infty} \left\{ b_n \cos\left(\frac{n\pi}{l}t\right) + \frac{d_n l}{n\pi} \sin\left(\frac{n\pi}{l}t\right) \right\} \sin\left(\frac{n\pi x}{l}\right)$$

Dabei sind b_n die Fourier–Koeffizienten der Entwicklung der vorgegeben Anfangsbedingung $u(x,0) = g(x)$ und d_n die entsprechenden Koeffizienten von $u_t(x,0) = h(x)$.

11. Numerical Methods for Elliptic Equations

Model Problem

$$-\Delta u = f \quad \text{in } \Omega = [0,1]^2 \in \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$



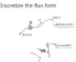
Finite Differences

Discretize the differential operator

$$-\Delta = L \approx L_h$$

Finite Volumes

Discretize the flux form



Finite Elements

"Discretize function space"

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$
$$\Rightarrow \int_\Omega -\Delta u \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi$$
$$\Rightarrow \int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi$$

Linear System of Equations

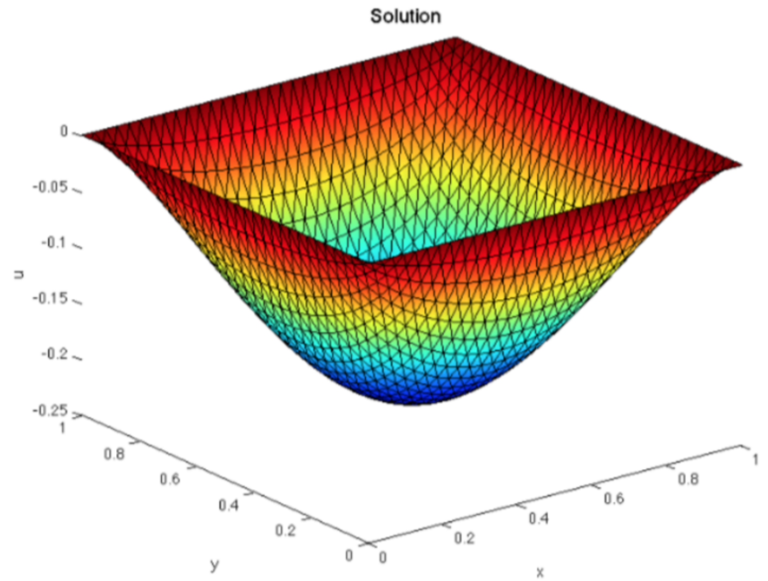
$$L_h u_h = f_h$$

rier Method

Model Problem

$$-\Delta u = f \qquad \text{in } \Omega = [0,1]^2 \in \mathbb{R}^2$$

$$u = 0 \qquad \text{on } \partial\Omega$$



Finite Differences

Discretize the differential operator

$$-\Delta = L \approx L_h$$

here: $$\frac{du}{dx} = \frac{u(x_{i+1}) - u(x_i)}{\Delta x} + \mathcal{O}(\Delta x), \quad \Delta x = x_{i+1} - x_i.$$

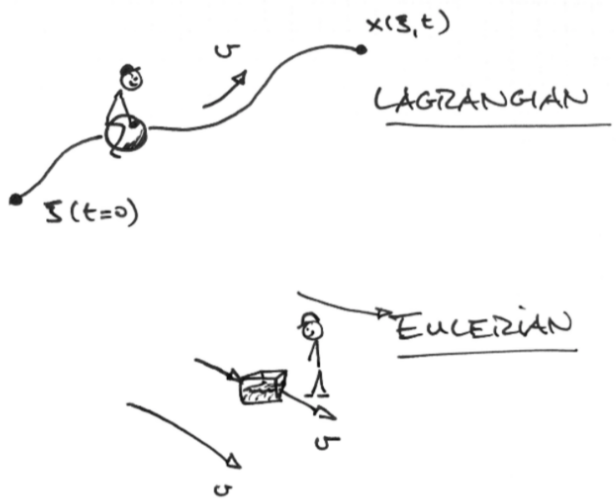
$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$-\Delta u = \frac{4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

Finite Volumes

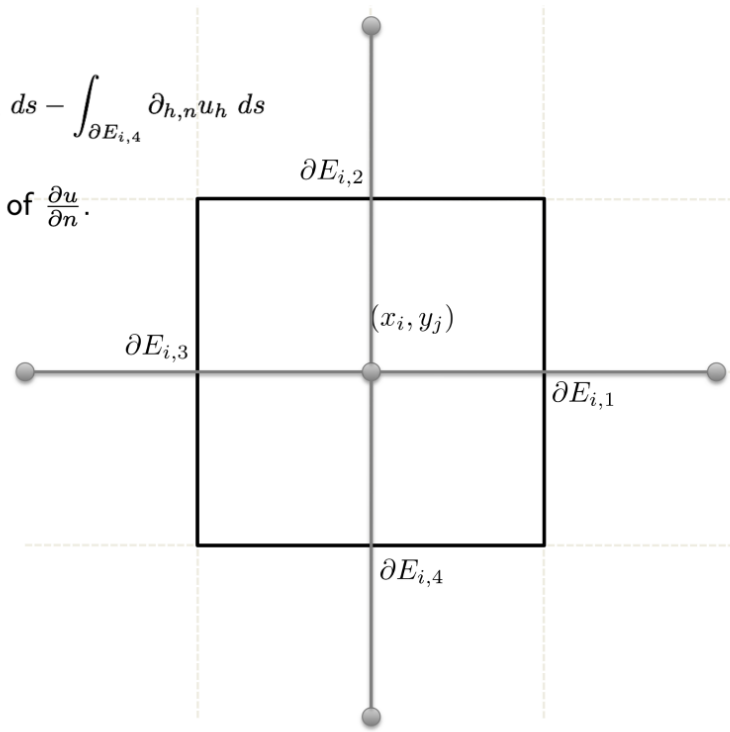
Discretize the flux form



$$\int_{E_i} f \; dx = -\int_{\partial E_i} \frac{\partial u}{\partial n} \; ds$$

$$\approx -\int_{\partial E_{i,1}} \partial_{h,n} u_h \; ds - \int_{\partial E_{i,2}} \partial_{h,n} u_h \; ds - \int_{\partial E_{i,3}} \partial_{h,n} u_h \; ds - \int_{\partial E_{i,4}} \partial_{h,n} u_h \; ds$$

$\partial_{h,n} u_h$: finite difference approximation of $\frac{\partial u}{\partial n}$.



$$\int_{\partial E_{i,1}} \partial_{h,n} u_h \; ds = \Delta x \cdot \left[\frac{u_h(x_{i+1}, y_j) - u_h(x_i, y_j)}{\Delta x} \right]$$

$$\int_{\partial E_{i,2}} \partial_{h,n} u_h \; ds = \Delta x \cdot \left[\frac{u_h(x_i, y_{j+1}) - u_h(x_i, y_j)}{\Delta x} \right]$$

$$\int_{\partial E_{i,3}} \partial_{h,n} u_h \; ds = \Delta x \cdot \left[\frac{u_h(x_{i-1}, y_j) - u_h(x_i, y_j)}{\Delta x} \right]$$

$$\int_{\partial E_{i,4}} \partial_{h,n} u_h \; ds = \Delta x \cdot \left[\frac{u_h(x_i, y_{j-1}) - u_h(x_i, y_j)}{\Delta x} \right]$$

Finite Elements

"Discretize function space"

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\Rightarrow \int_\Omega -\Delta u \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi$$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi$$

Now, replace the variational problem $a(u_h, v_h) = f(v_h)$ by

$$u_i \cdot a(\varphi_i, \varphi_j) = f(\varphi_j), \quad \forall i, j = 1, \ldots, N,$$

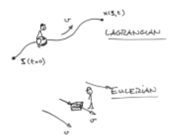
since $u_h = \sum u_i \varphi_i$.

Linear System of Equations

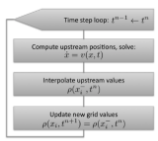
$$L_h u_h = f_h$$

12. Numerical Methodes for Transport Equation

Lagrangian Perspective



Algorithm

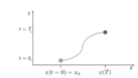


- Position: $x = x(t)$.
- Velocity: $v = v(x,t)$.

Particle position can be computed by

$$\dot{x} = \frac{dx}{dt} = v(x,t)$$

With initial condition $x(t=0) = x_0$



Problem: Passive advection ($s \equiv 0$):

$$\begin{aligned} \frac{dx}{dt} &= v(x,t), \quad x(0) = x_0, \\ \frac{d\rho}{dt} &= 0, \quad \rho(x,0) = \rho_0(x). \end{aligned}$$

Strategy:

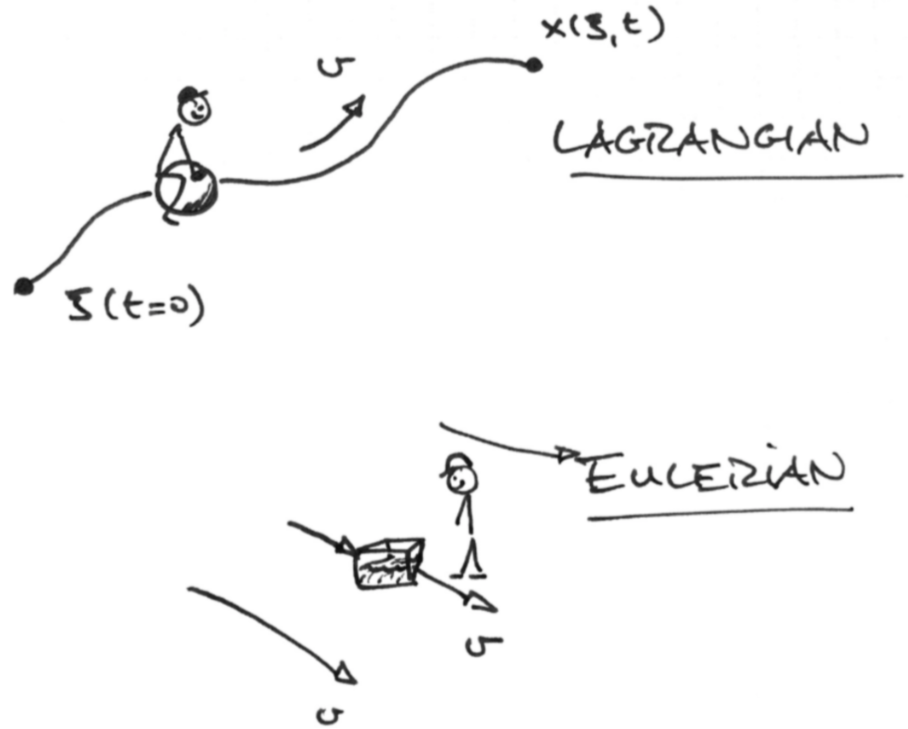
- Solve $\frac{dx}{dt} = v$ by any ODE solver,
- Solve $\frac{d\rho}{dt} = 0$ by finite difference.

$$\frac{d\rho}{dt} \approx \frac{\rho(x_i, t^{n+1} - \rho(x_i^-, t^n)}{\Delta t} = 0$$

$$\Rightarrow \rho^+ = \rho^-.$$

x_i, $i = 1:N$ grid points, t^n, $n = 1:M$ time steps.

Lagrangian Perspective



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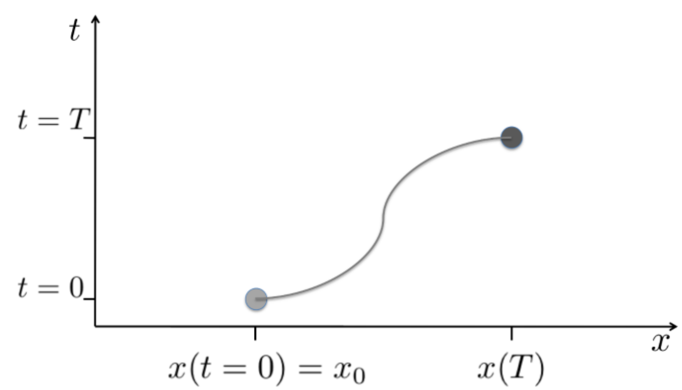

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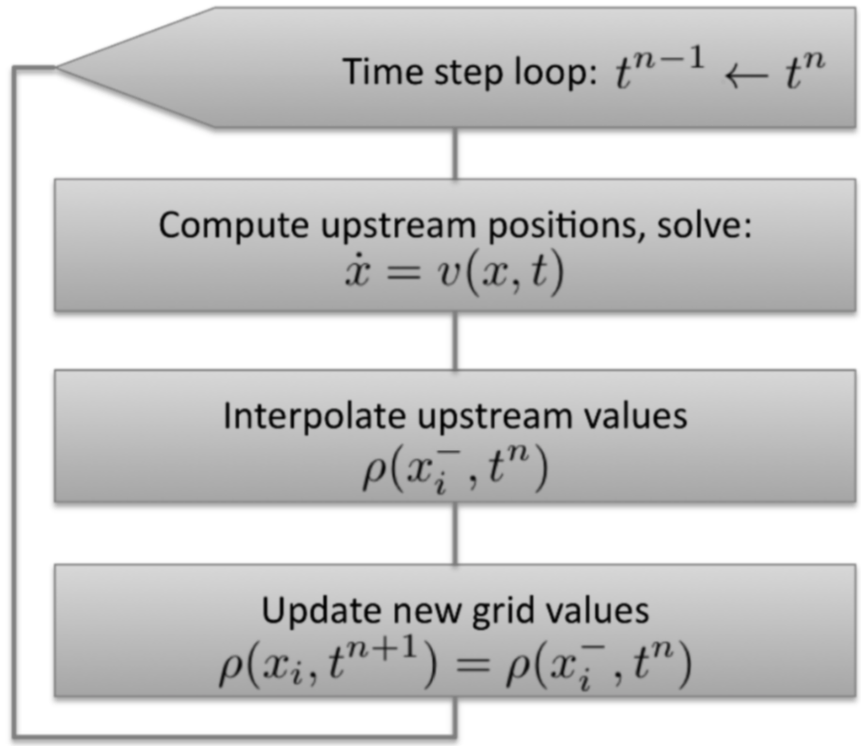


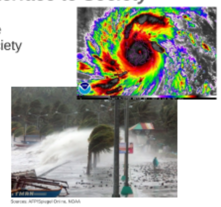
Time step loop: $t^{n-1} \leftarrow t^n$
Compute upstream positions, solve:
$\dot{x} = v(x, t)$
Interpolate upstream values
$\rho(x_i^-, t^n)$
Update new grid values
$\rho(x_i, t^{n+1}) = \rho(x_i^-, t^n)$

Society and Partial Differential Equations

Natural Hazards form the Interface to Society

Gobal Change
Impact on Society

Prevention
Mitigation
Planning



Deterministic Approach

- Uncerstanding the physical principle
- Physical model for probabilistic methods
- Solve differential equations!

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{F}(\rho) + \nabla(\nu \nabla \rho) = S(\rho)$$

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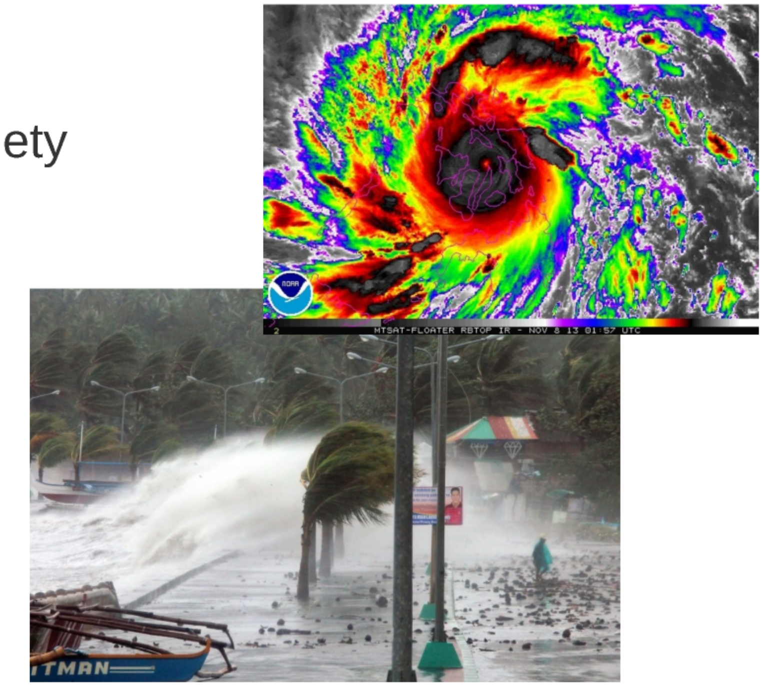

Sources: AFP/Spiegel Online, NOAA

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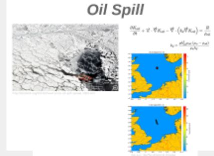
Examples

Tsunami

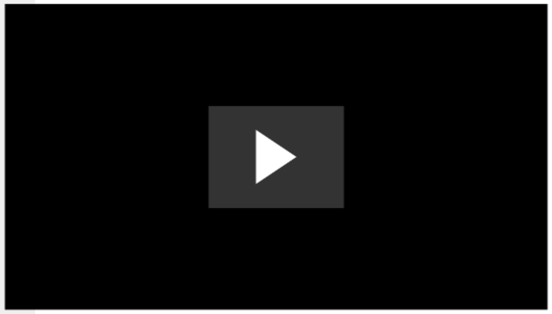
Storm Surge

Volcano Ash Dispersion

Oil Spill



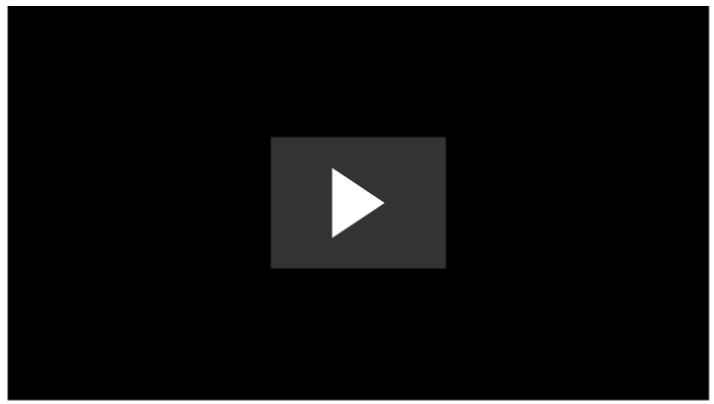
Tsunami



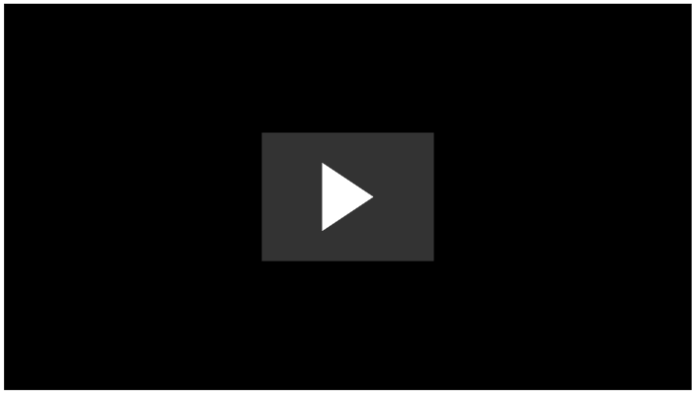
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + g\nabla\eta = R,$$
$$\frac{\partial \eta}{\partial t} + \nabla \cdot (H\mathbf{v}) = 0.$$

$$R = -f\mathbf{k} \times \mathbf{v} - rH^{-1}\mathbf{v}|\mathbf{v}| + H^{-1}\nabla(K_h H \nabla \mathbf{v})$$

Terms:
- Coriolis
- Bottom friction
- Viscosity (Smagorinsky approach)



Storm Surge



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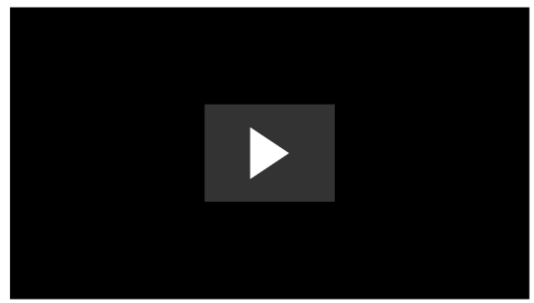
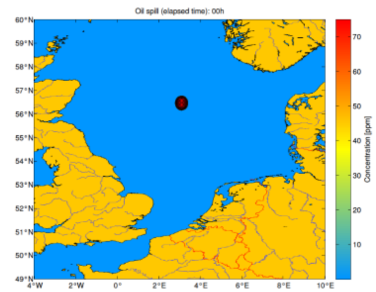


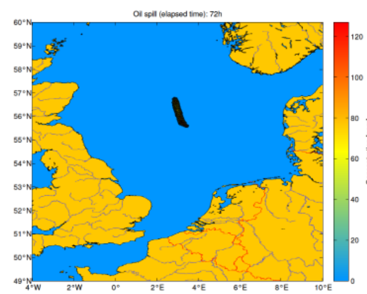
Oil Spill

http://tcktcktck.org/2014/07/wwf-arctic-oil-spill-spread-1000-km/

$$\frac{\partial K_{oil}}{\partial t} + \vec{u} \cdot \vec{\nabla} K_{oil} - \vec{\nabla} \cdot \left(k_d \vec{\nabla} K_{oil} \right) = \frac{R}{\rho_{oil}}$$

$$k_d = \frac{g h_{oil}^2 \rho_{oil} \left(\rho_w - \rho_{oil} \right)}{\rho_w k_f}$$



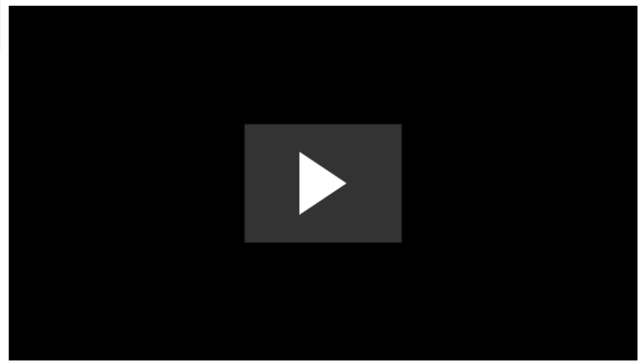


Source: Steffen Reinert (Study)

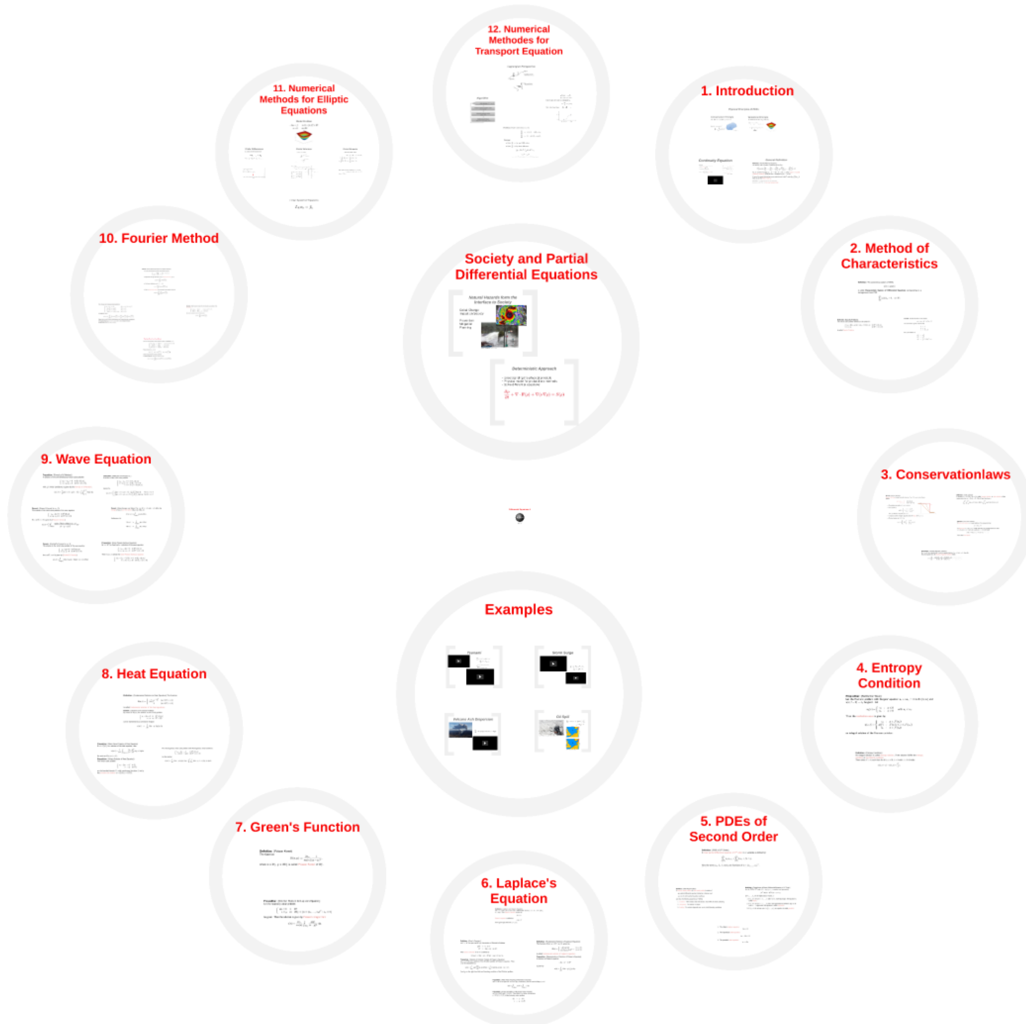
Volcano Ash Dispersion

Source: http://blog.pikarl.de/wp-content/uploads/2010/05/ls102139.jpg

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) + \nabla(\nu \nabla \rho) = S(\rho)$$



Source: Elena Gerwing, Matthias Hort, J.B. (MSc Project)


Society and Partial Differential Equations
1. Introduction
2. Method of Characteristics
3. Conservationlaws
4. Entropy Condition
5. PDEs of Second Order
6. Laplace's Equation
7. Green's Function
8. Heat Equation
9. Wave Equation
10. Fourier Method
11. Numerical Methods for Elliptic Equations
12. Numerical Methodes for Transport Equation
Examples