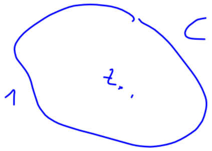
V0 Kompl. Fktn. 4.6.10

$$\oint_C (z-z_0)^n \, dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$



Cauchy Integralsatz

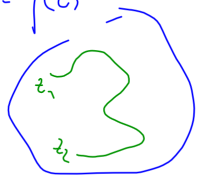
$$\oint_C f(z)\, dz = 0$$

f kompl. diff

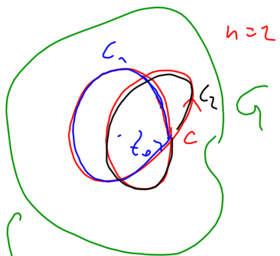
G e.z.h.

$\exists\, F(z)$ Stammfkt. $\quad F'(z) = f(z)$

$$\int_C f(z)\, dz = F(z_1) - F(z_2)$$

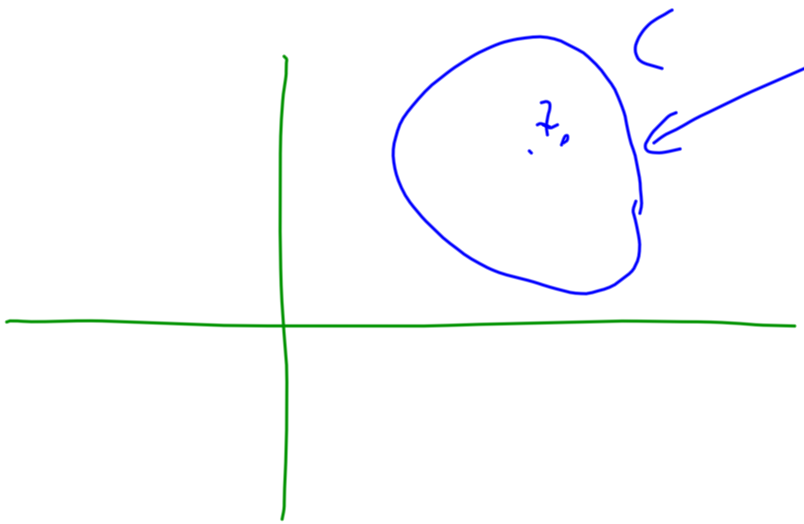


$$\frac{1}{2\pi i} \oint_C \frac{1}{z-z_0}\, dz = n$$



$$\int_C = \int_{C_1} + \int_{C_2}$$

In $\mathbb{R}^2$



f auf C bekannt

$f(z_0) = ?$

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz$$

$f(z) = 1$; ✓

$f(z) = z$

$$\boxed{\frac{1}{2\pi i} \oint \frac{z}{z - z_0}\, dz} = \frac{1}{2\pi i}\left(\oint \frac{\cancel{z - z_0}}{\cancel{z - z_0}}\, dz + \oint \frac{z_0}{z - z_0}\, dz \right)$$

$$= \frac{1}{2\pi i}\left(\underbrace{\oint 1\, dz}_{=0} \quad z_0 \underbrace{\oint \frac{1}{z - z_0}\, dz}_{2\pi i} \right)$$

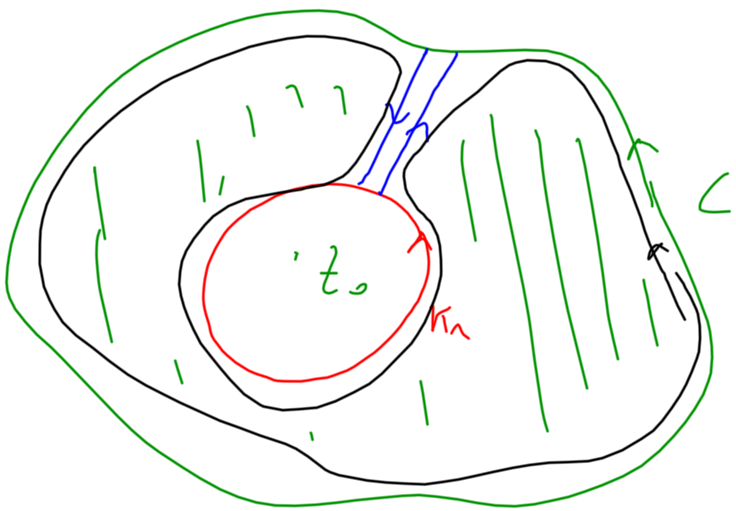
$$= \boxed{z_0}$$

$$f(z) = \frac{1}{z - z_0}$$

$$\frac{1}{2\pi i} \oint \frac{\frac{1}{z - z_0}}{z - z_0}\, dz = \frac{1}{2\pi i} \oint \frac{1}{(z - z_0)^2}\, dz = 0$$

$$\neq f(z_0) = \infty$$

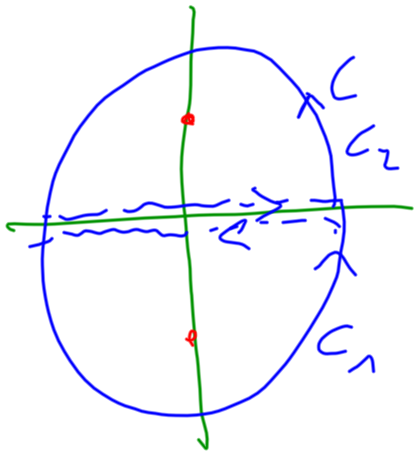
$f(z)$ in z_0 <u>nicht</u> diffb.



$$0 = \int_C \underbrace{\frac{f(z)}{z-z_0}}_{\text{diff'ni}} dz = \int_C + \int_C + \int_{-C} + \int_{-K_n} =$$

$$= \int_C + \cancel{\int_C} - \cancel{\int_C} - \int_{K_n}$$

$$\oint_C \frac{f(z)}{z-z_0} dz = \oint_C \frac{f(z)\,dz}{z-z_0}$$

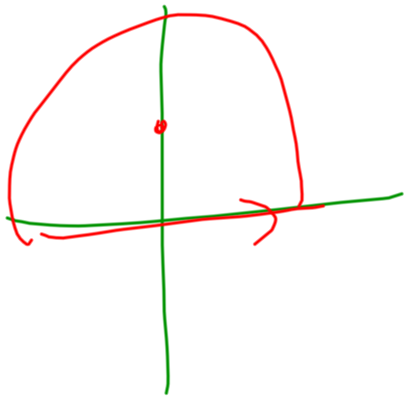


$$\oint_C \frac{1}{1+z^2}\,dz = (*)$$

$$\oint_C = \oint_{C_1} + \oint_{C_2}$$

$$(*) = 2\pi i \frac{1}{i+i} + 2\pi i \frac{1}{-i-i} = 0 \quad !!$$

$$\left[= \oint_{C_2} \frac{\left(\frac{1}{z+i}\right)}{z-i}\,dz + \dots \right]$$



$$f(z) = \frac{1}{z-i}$$

$$0 = \oint \frac{1}{(z-i)^2}\,dz = \oint \frac{\frac{1}{z-i}}{(z-i)}\,dz = f(z_0)\,2$$

$$z_0 = i$$

$$\overset{!}{=} \frac{1}{i-i} \overset{!}{=} \infty \quad \lightning$$

$\frac{1}{z-i}$ in i nicht kompl. diffb.

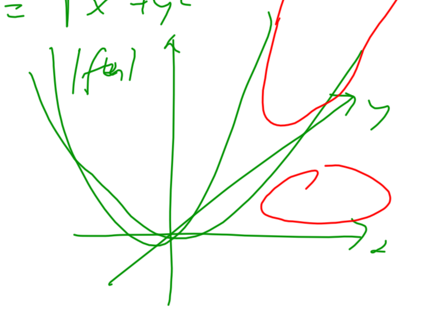
$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt$$

$$\boxed{|f(z)| = \frac{1}{2\pi}\left|\int \quad \right| \le \frac{1}{2\pi}\int | \quad |\,dt = \frac{1}{2\pi}\int |f(z_0+re^{it})|\,dt}$$

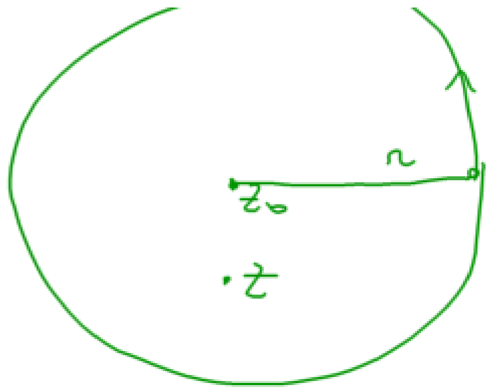
$$M = |f(z_0)| \qquad M \le \frac{1}{2\pi}\int |f(z_0+re^{it})|\,dt$$

$$\Rightarrow f(z_0+re^{it}) \stackrel{!}{=} M$$

$$f(z) = z \qquad |f(z)| = \sqrt{x^2+y^2}$$



$$f(z) = e^z \qquad |f(z)| = |e^{x+iy}| = e^x \quad \checkmark$$



Beweis Teil 4):

Nach der Cauchyschen Integralformel gilt

$$f(z_0) = \frac{1}{2\pi i} \oint\limits_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta$$

wobei der Kreis $|\zeta - z_0| = r$ einmal im positiven Sinn durchlaufen wird. Liegt nun z innerhalb dieses Kreises, d.h. $|z - z_0| < r$, so folgt ebenfalls

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Wir schreiben nun

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0} \cdot \frac{\zeta - z_0}{(\zeta - z_0) - (z - z_0)} \\ &= \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} \end{aligned}$$